

 HEWLETT-PACKARD CALCULATOR

Model 9100A

Training Manual



CALCULATOR TRAINING MANUAL

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CHAPTER I

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The -hp- Calculator was developed in the Corporate -hp- Laboratories in conjunction with Loveland Division engineers. The Calculator represents a combination of three basic ideas--a new type of memory for storing constants or subroutines, a new set of algorithms or logic used to define the set of trigonometric functions, and a unique electronic organization or logic.

This new type of memory is a read-only memory, abbreviated ROM, and is used for storing the keyboard subroutines. The ROM gives us a major competitive advantage because it enables us to store a large number of information bits (32,000) in a relatively small box and at a much lower cost than could be done using conventional memory systems. A read-only memory constructed from diodes would require about 10,000 diodes, be in a separate box about the size of the Calculator, and increase the selling price by \$2,000 - \$3,000.

ROM

Reliability is also a feature of the -hp- read-only memory. It is an entirely passive device; and once it has been tested at the factory, it should last for the life of the instrument. There are no diodes to fail and no magnetic levels to change. This, of course, is quite a customer benefit. Until our competitors develop a similar type ROM, they will not be able to offer a calculator that will directly compete with the 9100A. Rather, our future competition may offer fewer keys but different key sets and allow their customers to choose the key set which best fits their need.

RELIABILITY

It is very important for us to capitalize on our existing competitive advantage. Olivetti and other firms which have not yet entered the calculator field are working on more advanced models. Therefore, we probably enjoy a greater competitive advantage at this time than we ever will in the future; and it is very important for us to penetrate those new calculator markets where the -hp- name is not well known. It will never be easier!

We, of course, are also working on more advanced models as we feel our objectives are far broader than the 9100A. Briefly stated, our objective is to make and maintain -hp- as a leading supplier of electronic calculators. Our immediate goal is to expand the peripheral equipment such as printers, plotters and interface boxes for the existing 9100A Calculator. Beyond that, we are working on the design of the calculators for the 1970's. We would very much like to receive your comments about what you would like to have to sell five years from now. Or rather, what you think the market will need in five years.

THE FUTURE

TYPE OF CUSTOMERS

Seventy percent of your calculator customers will be people who have not previously talked with an -hp- field engineers. Of these, one-half will work for companies -hp- has not called on. Two significant points stand out with regard to our present customers:

We will not learn of their needs in the normal test instrument call.

In some cases this represents a substantial increase in the amount of time required to visit the additional departments. For example, at Chrysler we call on only 2 of the 28 engineering groups who might buy calculators.

The fact that 35% of the customers are from companies or businesses that we do not call on at all, further demonstrates the need for additional coverage.

BUYING HABITS

As many as 20% of your customers will desire to lease rather than purchase. The civil engineer usually always leases his equipment. You should be familiar with the current -hp- lease plans.

The majority of people who now own calculators also have a service contract. This appears to be a strong carry-over from the days of mechanical calculators and could change as they realize that an electronic calculator is more reliable. Most service contracts are sold for 6 - 7% of the cost of the equipment. Generally the customer expects faster service on his calculator than on his test equipment. He may have several voltmeters and so could get by without one, but he probably has only one calculator and depends on it.

GENERAL COMMENTS

Initially almost all sales will require a demonstration. A good demo takes about 30 minutes followed by 15-20 minutes for customer questions and solving specific customer applications.

The degree of research and engineering as measured by both dollars and number of people is a good indicator of how many calculators a firm might buy.

The specifier in a company is usually appointed because of his background and experience rather than position. In some cases this may be the computer center, accounting department or even the department manager. Generally one individual is considered the authority on calculators and most people are aware of who he is.

COMPUTER CENTER

People from the computer center are not good contacts. There are several characteristics which often show up.

They may be defensive about any device which will take work from them. Even if they like it, they feel it should be in their area.

Usually, they believe that larger computers are the best way to go. Computer center personnel have a vested interest in pushing a trend to buy larger computers rather than smaller.

They don't appreciate or understand why we consider our calculator easier to program; after all, everyone knows Fortran!

Most computer people feel that the large computer is available to anyone who wants it and since the computer is a fixed expense, it is more economical than a calculator. They don't like the idea of engineers writing inefficient programs, maybe getting one-half way through and then needing help. Of course, the engineers in their labs don't share these feelings. Most engineers complain about long turnaround times and programmers who don't understand their problems. For example, at one account, the director of the computer center told us that they would never buy a calculator for their labs because their 6500 was available to everyone. Yet, less than a month later, IME sold their engineers a calculator.

People working with large computers generally consider the calculator a toy.

They are not aware of the type of "calculator" problems that actually exist in their engineering labs, and don't have much influence over the labs they serve.

The computer people do use calculators for testing algorithms and checking programs. They are potential customers, and many will buy calculators. But, they are not apt to give you good leads to other departments.

Most groups are aware of Time Share systems, but believe that they cost only \$100/month. This is true of even the companies that have one. In reality most terminals cost \$700-\$1,000/month.

TIME
SHARE

We found very few users of either Programmable Calculators or Time Share that made extensive use of fixed program libraries. Generally they had all modified the programs for their applications or in most areas written their own. You will have to be a proficient programmer to take maximum advantage of our library by modifying it for specific customer applications.

Customers in the electronics industry seem to be able to make the fullest use of our keyboard subroutines. Of course, this is the area in which our test equipment sales give us the most contact. In addition, our present customers in this area are also better prospective customers for calculators than customers in other fields. In the mechanical field, for example, users of electronic instrumentation are not normally also users of calculators.

ELECTRONICS

Although most electronics companies have bought service contracts for the presently owned calculators, several doubted their economics. Our electronic customers will probably represent the largest group that does not buy service contracts.

CHEMISTRY

The analytical chemist will not be a major customer for the 9100A. He usually has simple calculations requiring a very simple calculator or a very large computer with lots of data storage. However, the chemical engineering field is very attractive. Many chemical engineering problems involve statistics, molecular weights, percentage concentration, etc. which are easily solved using the 9100A.

TRANSPORTATION EQUIPMENT

Companies in the transportation equipment business make extensive use of computer terminals. You should prime yourself on the relative place of desk calculators and time sharing before making a big pitch to automobile or airframe manufacturers. This area may require a sizeable increase in the number of calls required to sell calculators. The mechanical departments have greater potential than the electrical ones. There are a number of small job shops related to the automotive industry that could use calculators.

AEROSPACE

Aerospace companies consolidate purchasing requirements with the purchasing department playing a large role in evaluation and negotiation. Many companies have an engineering facility group which is very helpful in getting you connected with specific engineering departments. A wide variety of engineering disciplines are represented. It is hard to get to see all the people interested in calculators. One technical group is not very helpful in referring you to the next group. While they have larger computers, perhaps more than any other industry, these computers are not readily accessible to the people with problems. So accessibility and availability are good selling tools for the calculator.

CONSULTING ENGINEER

Consulting engineers seem to have the greatest need for a formatted output. They typically use a lot of data and need a permanent record. Also, there are special purpose computers available to them with a formatted output.

This is probably the easiest group to reach by direct mail because most civil engineers are registered. Consulting engineers can make the most effective use of a prepared program library. Their problems tend to use the same equations over and over; more so than any other area. They will require a lot of hand holding. At least they seem to have problems with the Calculators/Computers they are now using. They expect and seem to get assistance from salesmen now selling them competitive equipment.

Since this group is found in smaller offices, the ratio of engineers per calculator is higher; but the opportunity for repeat sales is small. Most of the smaller offices have ties with a computer center either in a university or a larger office. So the fact that we can't solve their larger problems is not necessarily a deterrent to a sale. A calculator for office use with the computer center for back-up, may be the best and most economical solution for them. Only one out of three engineers in a consulting engineer's office do the type of work that requires a calculator. Others are involved in proposals, drafting, etc.

Contractors who buy calculators are usually larger firms. The smaller firms usually do not do their own engineering. The data processing man is the primary contact with contractors and is more receptive here than in most places.

University professors do a lot of funded research which requires use of calculators. Since very little equipment is shared, the ratio of calculators to personnel will be higher in this area. There are usually 10-15 departments in each school with a potential of several units each. You will usually deal directly with the end user who is also the specifier of calculators. He commonly has money set aside in a grant for computations. This money can be spent to buy calculators. Student problem solving is another big university application. Several schools have calculators available in a central area for students.

The following questions illustrate good answers to the most common questions of a casual nature.

- Q. What features does your calculator have that your nearest competition lacks?
- A. Olivetti--Speed, dynamic range, total addressing capabilities, ability to correct program steps, 175 decades more dynamic range, transcendental functions, simple programming language.
- Wang -- Speed, unlimited qualifiers, mag card, 3 register display, completeness and accuracy of transcendental functions, self contained unit.
- Small computers -- Less expensive, simple man-machine interface.
- Q. What is the difference between this machine and a computer?
- A. Traditionally, a computer has not had an input console which contains the English and algebraic keys as is found on the calculator. Input to a computer is not keyed in by the user--it is prepunched on tape or cards. The fact that we have a device that will allow a human in the control loop is the thing that sets a calculator apart from a computer.

CONTRACTORS

UNIVERSITIES

COMPETITION

COMPUTER

HARDWARE

INTEGRATED CIRCUITS

We feel that it has the physical features normally attributed to a calculator even though its insides are structured like a computer; most people identify it by its physical features and not by its operating characteristics.

- Q. Compare your calculator to a time share terminal.
- A. The calculator and time share can stand on their individual merits. We offer portable, silent operation and a sense of "total ownership" that is unattainable on a time share terminal.

Most everyday operations are performed much more efficiently on the calculator than on time share--finding the arc tan of a number requires 125 operations and six minutes on time share, the calculator requires 2 operations and milliseconds. We offer an accuracy that is not characteristic of time share.

- Q. List the ways in which your 9100A Calculator will outperform some computers.
- A. 1. Portability, size, cost
2. Ease of operation
3. Convenient man/machine interface
4. Accuracy, broad dynamic range
5. It is unnecessary to learn a special computing language.
- Q. I understand that you have a read-only memory. Would you comment on this.
- A. The know how in the ROM design and manufacture we consider proprietary.
- Q. Do you use multilayer boards in the calculator?
- A. Yes, the conventional "back panel" wiring is a multilayer board.
- Q. Do you use epoxy transistors?
- A. We use both metal and epoxy case transistors.
- Q. Do you manufacture your own?
- A. Keys YES
Case IT IS CAST ELSEWHERE
CRT YES
Multilayer boards YES

- Q. I understand that you did not elect to use integrated circuits. Why?

- A. Much of the circuitry such as core drivers conduct currents that exceed the upper limits offered by I. C. 's. We use unique circuitry not currently available in I. C. packs.

The power requirements of the IC's considerably exceeds that of the discrete circuitry used. The power saving shows up in two ways--a smaller power supply and no fan is required. The package would be considerably larger if IC's were used.

Q. Will you redesign the calculator to use in-house IC technology?

A. When IC technology allows us to obtain a substantial savings without loss of performance we will consider a redesign.

Q. How does the cost divide up in the system?

A. We had originally estimated that the core, logic (including ROM), and supporting hardware (case, keyboard, CRT, power supply, etc.) would each share 1/3 of the cost. It turns out that this estimate held true.

Q. How many man years went into developing the calculator?

A. 700 man months over a 2 1/2 year period.

Q. Will you redesign the calculator for special purposes--for instance, the hyperbolic functions may be of limited value to civil engineers, while degree, minute, second-to-decimal conversions would have a great appeal. Can you accomodate such changes?

A. We can accomodate such changes. However, they usually represent a major effort and the merits of each must be thoroughly evaluated.

Q. Can you make a stripped down version?

A. Yes, but the selling cost will not decrease as rapidly as the loss of calculating power.

Q. Will you offer the calculator or parts of it as an OEM item to systems house?

A. We would certainly like to discuss such applications in detail.

Q. Will you lease the machines or offer lease/buy contracts.

A. Yes, there are a number of lease options.

Q. You have no printing mechanism. Isn't that a serious liability?

A. No, many applications do not require a printer. However, for those who have a printing requirement, a small silent printer is in development and will be available in 1969.

Q. Have you any other peripherals in the offering?

A. We are working on an X-Y recorder for plotting functions. We have an input/output connector in the back which allows I/O devices to be connected to the calculator. The PRINT and FORMAT keys are specifically designed for I/O equipment.

Q. You have mentioned a printer and an X-Y recorder as peripherals. How much will they cost?

A. Prices have not been established. We estimate they will be in the neighborhood of \$1,000 and \$2,500 respectively.

SPECIALS

OEM

LEASE

PERIPHERALS

ACCURACY

Q. What is the accuracy of the machine?

A. Many of the functions are non linear in nature and require graphs or tables to extract the absolute error. However, in almost every case the calculated $f(x)$ is bounded by the true values for $f(x)$ as x is allowed to vary between plus or minus one count in the tenth significant digit.

DYNAMIC RANGE

Q. You have a 10 digit display. Some of your competition has 15 or 20 digits. Isn't this a serious liability.

A. Most of the competitive calculators do not operate as true floating point machines. (There is some confusion between automatic decimal positioning and floating point.) The dynamic range of a non-floating point machine is limited to the register size. If one considers all problems, it can be shown that the effective register length (the length at which no loss of accuracy occurs) on a non-floating machine is limited to one-half of the register length.

Q. Why in the world do you need such a broad dynamic range?

A. Although the factors associated to almost all physical phenomena are well within the dynamic range of the calculator, some intermediate products and quotients involve very large numbers which can overflow machines having a narrow dynamic range.

TRAINING

Q. With all of the complex functions, it must require a considerable training period to master the device. How long does it require to become proficient?

A. A prime feature is the ease with which one can learn to use the instrument. The basic calculator operations (+, -, x, ÷, etc.) can be mastered in minutes. The same applies to the complex functions, provided the user understands what the function means. If the user has had any experience in computer programming, then an hour is adequate for learning the programming commands. If he has never programmed, then it requires a few hours to learn the concepts of looping, branching, initializing, etc.

Q. What types of training manuals will be used?

A. The instrument will be supplied with all necessary training and operating supplies:

- A comprehensive operating and programming manual
- A quick review pull out card attached to the instrument
- A comprehensive library containing ready to use programs and program techniques
- Programming cards

Q. Will you maintain a programming service?

A. We will maintain and distribute programs submitted by users or developed at Hewlett-Packard. A regular publication, THE KEYBOARD, is sent to all users to keep them informed of new programs and applications.

Q. How much will additional program cards and coding sheets cost?

A. The following are furnished with the calculator; replacements are available at:

Program Library	\$30.00
Operating Manual	2.50
Program Pads	10.00
(5 pack - 500 sheets)	
10 Mag Cards in Flip Top Box	10.00
Diagnostic Card	2.50
Dust Cover	5.00

Q. How will the instrument be serviced?

A. Each user is given a diagnostic program which automatically tests each operation that can be executed. If this fails to run, the machine will be serviced on a major subassembly replacement basis. If the customer has a service contract, the calculator will be serviced on his premises; if there is no service contract, it will be serviced by returning it to an -hp- service center.

SERVICE

Q. What do you expect for a mean-time-between-failure?

A. We are offering a standard one year warranty, which is considerably longer than the typical 90 day warranty offered by the competition. This is made possible by taking such measures as preageing each transistor for 100 hours and testing it before use in the calculator.

Q. Do you offer a service contract after the 1 year warranty?

A. Yes.

Q. Can you ruin the machine by pressing the keys in a random order?

RELIABILITY

A. No, you may set it to calculating a problem that will take eons for a solution; however, it can always be stopped by pressing the STOP key.

Q. How can I tell if it is making a mistake?

A. If any of its active circuitry is out of commission, it will almost invariably show up by an odd looking display. The diagnostic program provided with each calculator will exercise and check all keys. It is doubtful that the diagnostic would run and yet have the machine working improperly.

Q. Does the calculator have error detecting or error correction circuitry?

ILLEGAL
OPERATIONS

A. We have a warning light that detects illegal mathematical operations--many calculating errors result from illegal operations. Hence, if the warning light lights during an operation, an error has occurred. If there is any question on reliability, the diagnostic routine should be run.

ENVIRONMENT

- Q. What are the environmental conditions under which the calculator will operate?
- A. It passes the environmental specifications of 0 to 55C, and 95% relative humidity at 40C.
- Q. Can the calculator be operated from a field battery source?
- A. It would require an inverter capable of delivering 70 watts.
- Q. How much abuse can a mag card withstand?
- A. Certainly it must not be placed in a strong magnetic field; however, I have rubbed the card several times over the calculator top without loss of information. Since most cards receive little abuse in their protective case, I can see little concern for losing information by rough handling. The cards are recorded in a NRZ saturated mode which offers excellent noise immunity.

CHAPTER II
K E Y B O A R D

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The Hewlett-Packard 9100A Calculator is a scientific desk-top calculator. Trigonometric and logarithmic functions are calculated with a single keystroke. The Model 9100A is programmable; conditional branch instructions provide complete programming capability. In addition, two programs can be recorded on each magnetic card supplied with the 9100A.

The Calculator has two memory systems. One is a unique Hewlett-Packard Read-Only-Memory. It contains the subroutines necessary to execute the keyboard instructions and to calculate the trigonometric and logarithmic functions.

The 9100A also contains a magnetic core memory which adds the storage and programming features. There are 19 accessible registers in the magnetic core memory; three are displayed, two are only for data storage. The 14 program registers can accommodate up to 196 program steps or can be used for additional data storage.

Three registers are displayed on the cathode ray tube; they are designated X KEYBOARD, Y ACCUMULATOR and Z TEMPORARY. The format of the display can be selected on the keyboard. With Floating Point notation, the contents of the registers are displayed as a ten-digit number and a two-digit exponent, indicating the power of ten multiplier. This display mode allows simultaneous presentation of very large (up to 10^{99}) and very small (down to 10^{-98}) numbers. With Fixed Point notation, the contents of the registers are displayed in decimal format, with the number of digits to the right of the decimal (0 to 9) set by the decimal thumbwheel.

A magnetic program card loaded with the Diagnostic Program is shipped inserted in the rear pocket of the Program Library. To enter the program:

GENERAL DESCRIPTION

DIAGNOSTIC PROGRAM

SET: ON

SET: RUN

SET: RADIANS

SET: FIXED

PRESS: GOTO 0 0 OR END

Be sure magnetic card is fully inserted in the card reader with the printed side of the card towards the user.

PRESS: ENTER

PRESS: CONTINUE

Execution of the program (indicated by a flashing display of $X = 3$, $Y = 2.000000$ and $Z = 1$) ensures that the 9100A is operating correctly.

ACCESSORIES

The accessories and equipment supplied with each Model 9100A are listed in Table 1.

TABLE 1
Accessories/Equipment Supplied

PART NO.	QUANTITY	DESCRIPTION
09100-90001	2	Operating and Programming Manual
09100-90002	1	Program Library
09100-90003	1	Program Pad
09100-90004	1	Magnetic Program Card Loaded with Diagnostic Program
4040 - 0350	1	Dust Cover
5060 - 5919	1	Magnetic Program Card Container with Ten Program Cards
8120 - 0078	1	Power Cord
9320 - 1157	1	Roll-out Instruction Card contained in the bottom of the 9100A

KEYBOARD

OFF ON

DEGREES RADIANS

PROGRAM RUN

FLOATING FIXED

Switches ac line power to the Calculator. Application of ac line power lights the register designators at the right of the cathode ray tube.

Selects the entry and display units for the trigonometric functions.

Selects the mode of operation.

RUN: used to perform calculations, address the program counter, execute a program, and enter the mag card.

PROGRAM: used to enter and verify a program manually.

Selects the mode of display

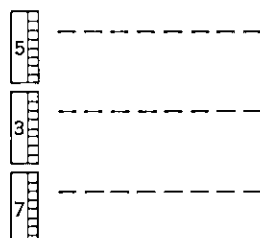
FLOATING DECIMAL POINT: the display consists of a ten-digit number and a two-digit exponent, indicating the power of ten multiplier.

NUMBER
 $12,345.67898 = 1.234567898 \times 10^4$

DISPLAY
1.234 567 898 04

FIXED DECIMAL POINT: the number is displayed with the decimal point correctly located. The decimal wheel sets the number of places to the right of the decimal point. The least significant digit is rounded. If the number overflows to the left of the decimal point, the display for the overflowed register reverts to floating decimal point.

DECIMAL WHEEL SETTING



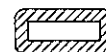
DISPLAY

12345.67898

12345.679
 Leading Zeros Blanked Last Digit Rounded

Number too large for this setting—automatically shifts to floating decimal point.

The error light indicates an illegal operation such as division by 0., $\sqrt{-X}$, $\sin^{-1}X$ ($|X| > 1$), $\ln X$ ($X < 0$). An illegal operation during program execution will not stop the program.



Clears the X, Y, Z, e and f registers (0. → X, Y, Z, e and f). Clears the FLAG and clears the ARC and HYPER conditions.



Clears the X register (0. → X). Clears the ARC and HYPER conditions. It is not necessary to press CLEAR X before each entry.



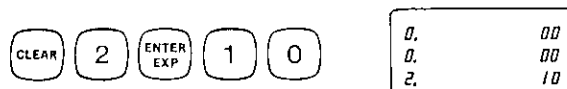
Changes the sign of the contents of the X register. Changes the sign of the exponent if ENTER EXP was pressed preceding CHG SIGN.



Sets the decimal point. It is not necessary to enter the decimal point when entering integers or when using the ENTER EXP key.



Clears the exponent and causes the next digit entries (0 to 99) and CHG SIGN to affect only the exponent. The exponent digits are entered in a serial manner, with each new digit entry becoming the least significant digit of the exponent.



Pressing ENTER EXP after any keyboard operation, except digit entry, will enter 1.0 for the number in X. To restore control to the mantissa, press the DECIMAL key.

DATA ENTRY

CLEAR ENTER EXP CHG SIGN 6

D.	00
D.	00
1.000 000 000-06	

All keyboard entries are displayed in the X register.

SET: FLOAT

ENTER: 126.78

CLEAR 1 2 6 . 7 8

D.	00
D.	00
1.267 8 02	

CLEAR 1 2 6 7 8 ENTER EXP 2

SET: FLOAT

ENTER: -39,800 = - 3.98 x 10⁴

CLEAR X CHG SIGN 3 9 8 0 0

D.	00
D.	00
-3.980 0 04	

CLEAR X 3 9 8 CHG SIGN ENTER EXP 4

SET: FIXED

SET: Decimal wheel to 4

CLEAR X 9 4 ENTER EXP 3 CHG SIGN

D.	
D.	
.0094	

CLEAR X . 0 0 9 4

SET: ☐ FIXED

SET: Decimal wheel to 4

ENTER: 128.0167

0.
0.
128.0167

SET: Decimal wheel to 3

0.
0.
128.017

SET: Decimal wheel to 8

0.
0.
1.280 167 02

SET: ☐ FLOAT

ENTER: -4×10^{-12}

0. 00
0. 00
-4. -12

These keys reposition the contents of the display registers as required for succeeding calculations. x, y and z refer to the contents of X, Y, and Z.

☐ FIXED

Duplicates the contents of X to the Y register and shifts the contents of Y to the Z register. Loses the contents of the Z register.

4.
3.
3.

Duplicates the contents of Z to the Y register and shifts the contents of the Y register to the X register. Loses the contents of the X register.

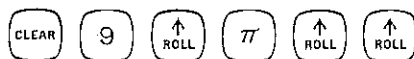
6.
6.
5.

DISPLAY REGISTERS





Shifts the contents of X to the Y register, shifts the contents of Y to the Z register and shifts the contents of Z to the X register.



3.1416
0.
9.



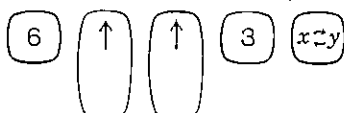
Shifts the contents of Z to the Y register, shifts the contents of Y to the X register and shifts the contents of X to the Z register.



11.
0.
2.



Exchanges the contents of the X and Y registers. The contents of Z are unchanged.



5.
3.
5.

ARITHMETIC KEYS



The arithmetic keys operate on the X and Y registers, entering the result of the calculation in the Y register and leaving the X register unchanged.

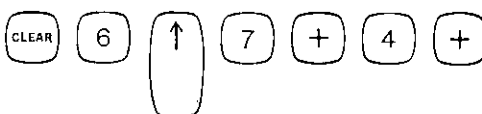
Adds the contents of the X register to the contents of the Y register. The sum appears in the Y register and the X register is unchanged.

$$8 + 4 = 12 \rightarrow Y$$



0.
12.
4.

$$6 + 7 + 4 = 17 \rightarrow Y$$



0.
17.
4.

Subtracts the contents of the X register from the contents of the Y register. The difference is entered into the Y register and the X register is unchanged.

—

$$17 - 9 = 8 \rightarrow Y$$

CLEAR 1 7 ↑ 9 —

0.
8.
9.

$$47 - 19 - 8 + 5 = 25 \rightarrow Y$$

CLEAR 4 7 ↑ 1 9 — 8 — 5 +

0.
25.
5

Multiplies the contents of the X register by the contents of the Y register. The product is entered into the Y register and the X register is unchanged.

×

$$9 \times 7 = 63 \rightarrow Y$$

CLEAR 9 ↑ 7 ×

0.
63.
7.

$$3 \times 5 \times 4 \div 8 = 7.5 \rightarrow Y$$

CLEAR 3 ↑ 5 × 4 × 8 ÷

0.
7.5000
8.

Divides the contents of the Y register by the contents of the X register. The quotient is entered into the Y register and the X register is unchanged.

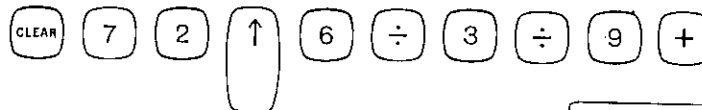
÷

$$36 \div 4 = 9 \rightarrow Y$$

CLEAR 3 6 ↑ 4 ÷

0.
9.0
4.

$$72 \div 6 \div 3 + 9 = 13 \rightarrow Y$$



0.
13.0.
9.

EXAMPLE
$$\frac{(3 \times 4) + (6 - 9)}{(8 \times 2) - 6} = .9 \rightarrow Y$$

STEP	KEY
1—	3
2—	↑
3—	4
4—	×
5—	6
6—	+

STEP	KEY
7—	9
8—	—
9—	8
10—	↑
11—	2
12—	×

STEP	KEY
13—	6
14—	—
15—	↓
16—	÷

STORAGE



The Storage and Recall Keys provide access to the 16 storage registers: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e or f. It is not necessary to clear a register before storing a number; the new number is written over the previously stored number.

Stores the contents of X in the register indicated by the following key-stroke: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, or f. The contents of the X register are unchanged.



Stores the contents of Y in the register indicated by the following key-stroke: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, or f. The contents of the Y register are unchanged.

RECALL

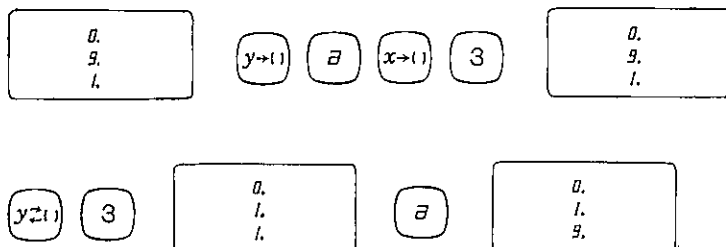


Press a, b, c, d, e or f to recall the contents of the register to the X register. The contents of a, b, c, d, e, or f are unchanged. The $y \rightarrow ()$ key must be used to recall the contents of the numeric registers. Pressing only 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 will not recall the contents of that register to the X register, but will enter that number in the X register.

Recalls the contents of f and e to the X and Y registers. The contents of f and e are unchanged.

Exchanges the contents of Y with the contents of the register indicated by the following keystroke: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, or f. This is the only instruction that recalls the contents of a numeric storage register.

$y \leftrightarrow ()$



Selects the entry and display units for the trigonometric functions. With DEGREES selected, an entry in degrees cannot be converted to radians by selecting RADIANS. Inverse trigonometric functions are calculated for the principal value of the function.

FUNCTION KEYS

- 0 $\sin^{-1}X$; -90° to $+90^\circ$ ($-\pi/2$ to $+\pi/2$ radians)
- 0 $\cos^{-1}X$; 0° to 180° (0 to $+\pi$ radians)
- 0 $\tan^{-1}X$; -90° to $+90^\circ$ ($-\pi/2$ to $+\pi/2$ radians)

EXAMPLE:

$$\sin 135^\circ = +.7071; \sin^{-1} +.7071 = 45^\circ$$

$$\tan -100^\circ = -5.671 \quad \tan^{-1} -5.671 = -80^\circ$$

Replaces the contents of the X register with the Sine of the contents of X.

$\sin x$

Replaces the contents of the X register with the Cosine of the contents of X.

$\cos x$

Replaces the contents of the X register with the Tangent of the contents of X.

$\tan x$

Prefix used to calculate the inverse trigonometric or hyperbolic functions. The answer is placed in the X register.

$\text{arc } x$

hyper
▼

Prefix used to calculate the hyperbolic (sinh, cosh, and tanh) functions. The answer is placed in the X register.

EXAMPLE: $\sinh^{-1} 2 = 1.4436$

CORRECT

2 arc hyper sin x

$\sinh^{-1} 2 \rightarrow X$

INCORRECT

2 hyper arc sin x

$\sin^{-1} 2 \rightarrow X$ ERROR
LIGHT

ln x

Replaces the contents of the X register with the logarithm to the base e of the contents of X.

e^x

Replaces the contents of the X register with e raised to the power defined by the contents of X.

log x

Replaces the contents of the X register with the logarithm to the base 10 of the contents of X.

$\sqrt{x}$

Replaces the contents of the X register with the square root of the contents of X.

Int x

Integer x. Eliminates the decimal part of the contents of the X register. Leaves the integer into the X register.

CLEAR 5 . 9 Int x

0.
0.
5.

|y|

Absolute value of Y. Sets the contents of the Y register positive.

CLEAR CHG SIGN 5 ↑ |y|

0.
5.
-5.

Enters pi in the X register.



Conversion from rectangular to polar coordinates will calculate the angle, θ , in the range:

$$-180^\circ < \theta \leq +180^\circ$$

$$-\pi \text{ radians} < \theta \leq +\pi \text{ radians}$$

VECTOR
KEYS

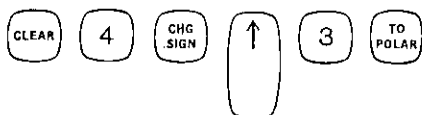
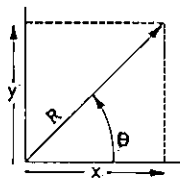
Changes rectangular coordinates consisting of an x component in the X register and a y component in the Y register to polar coordinates.



$$\text{Angle } (\theta) = \tan^{-1} y/x \rightarrow Y$$

$$\text{Radius } (R) = \sqrt{x^2 + y^2} \rightarrow X$$

SET: DEGREES



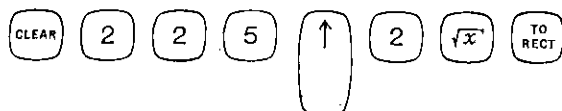
0.
-53.1301
5.

Changes polar coordinates consisting of a radius (R) in the X register at an angle (θ) in the Y register to rectangular coordinates:



$$y \text{ component} = R \cdot \sin \theta \rightarrow Y$$

$$x \text{ component} = R \cdot \cos \theta \rightarrow X$$



0.
-1.0000
-1.0000

ACCUMULATE +: Adds the contents of the X and Y registers to the contents of the f and e registers respectively. The sums are entered in the f and e registers. f and e are cleared with the CLEAR key.



$$f + X \rightarrow f$$

$$e + Y \rightarrow e$$

ACC
--

RCL

ACCUMULATE -: Subtracts the contents of the X and Y registers from the contents of the f and e registers respectively. The differences are entered in the f and e registers.

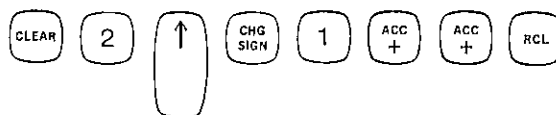
$$f - X \rightarrow f$$

$$e - Y \rightarrow e$$

Recalls the contents of the f and e registers to the X and Y registers respectively. The contents of f and e are unchanged.

$$f \rightarrow X$$

$$e \rightarrow Y$$



0.
4.
-2.

ACC -
RCL

0.
0.
0.

EXAMPLE: Vector Addition

$$(2x + 3y) + (4x + 5y) - (3x - 6y) = 3x + 14y$$

STEP	KEY
1—	CLEAR
2—	3
3—	↑
4—	2
5—	ACC +

STEP	KEY
6—	5
7—	↑
8—	4
9—	ACC +
10—	6

STEP	KEY
11—	CHG SIGN
12—	↑
13—	3
14—	ACC -
15—	RCL

EXAMPLE: Multiplication of complex numbers ($j^2 = -1$)

$$(3 + j4)(-2 + j3) = -18 + j1$$

STEP	KEY
1—	CLEAR
2—	4
3—	↑
4—	3
5—	TO POLAR
6—	In x

STEP	KEY
7—	ACC +
8—	3
9—	↑
10—	2
11—	CHG SIGN
12—	TO POLAR

STEP	KEY
13—	In x
14—	ACC +
15—	RCL
16—	e^x
17—	TO RECT

Range of arguments for trigonometric, hyperbolic, logarithmic and exponential functions.

RANGE OF ARGUMENTS

$\sin X, \cos X, \tan X;$	$ X \leq 1 \times 10^{10}$
$\sin^{-1} X;$	$ X \leq 1$
$\cos^{-1} X;$	$ X \leq 1$
$\tan^{-1} X;$	any X
$\sinh X;$	$ X < 230.25$
$\cosh X;$	$ X < 230.25$
$\tanh X;$	$ X < 230.25$
$\sinh^{-1} X;$	any X
$\cosh^{-1} X;$	$ X \geq 1$
$\tanh^{-1} X;$	$ X < 1$
$\ln X;$	$X > 0$
$\log X;$	$X > 0$
e^X	$-227.95 < X < 230.25$

Range of the 9100A: 1×10^{99} to 1×10^{-98}

Even though $1 + 10^{-99}$ may be entered and stored in the calculator, the calculator's lower range is 1×10^{-98} . To demonstrate this enter 2×10^{-50} and 5×10^{-50} into the X and Y registers and multiply. The Y register will overflow and read zero.

	Range	Average Time
1. + or -	2ms	
2. MPY	2-22ms	12ms
3. DIV	4-27ms	18ms
4. SQRT	4-20ms	19ms
5. SIN, COS, TAN	240ms (1st quad.) 330ms (1 circle)	280ms
6. ARC SIN, ARC COS	140ms	
7. ARC TAN	80ms	
8. EXP	110ms	
9. LN	50ms	
10. TO RECT	280ms	
11. TO POLAR	200ms	

CALCULATOR FUNCTION TIMES

CALCULATOR ERROR

A solution may be in error for the following reasons:

1. Approximate nature of physical law or principle upon which the solution is based.
2. Truncation error--this is caused by using a closed form expression of an infinite series such as:

$$\sin X \approx X - \frac{X^3}{3!} + \frac{X^5}{5!} - \frac{X^7}{7!}$$

Here we have truncated all powers of X higher than X^7 .

3. Round-off errors--causes error in numerical computation due to the use of finite number of significant digits. This is the limiting factor of the 9100A.
4. Mathematical errors in computation.

The accuracy of algorithms used in the Calculator is limited to the number of significant digits inputted and used in each calculation. The displayed digits, guard digits, and exponent digits give 12 significant digits and two digits for the exponent. Usually analysts are interested in magnitudes of numbers and the first few significant figures. Thus you can readily see the advantage of our scientific notation (floating point) and carrying 12 significant digits and exponent over competitive fixed point machines. One quickly recognizes magnitudes by looking at the exponent... and the added advantage of a range 10^{-98} to 10^{99} .

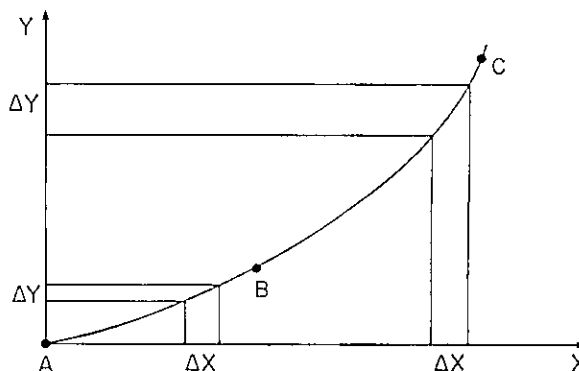
CALCULATION ERROR

The calculation error may be briefly explained in the following manner.

The two guard digits are carried along in all calculations extending the accuracy of the Calculator. In fixed display the least significant displayed digit is rounded. In floating point the least significant displayed digit is not rounded.

FUNCTION ERROR

The transcendental function error may be explained in the following way. Consider the function shown below



Let ΔX represent the smallest increment or the limit of the Calculator's resolution to describe the independent variable X . In the Calculator this would be the 12th digit or least significant guard digit.

Assume the value of the function in the region A to B is desired. A ΔX change causes a small change ΔY . However, if the value of the function in the region B to C is desired, a ΔX change causes a much larger change ΔY . The size of the error region, ΔY , will increase as the average slope of the function in that region increases. If the function rises exponentially, the error will increase quite rapidly as X increases.

The function error may be summarized as follows:

The calculated estimate, $\hat{f}(x)$, lies between the exact value of $f(X + \Delta X)$ and the exact value of $f(X - \Delta X)$; stated mathematically:

$$f(X - \Delta X) \leq \hat{f}(X) \leq f(X + \Delta X)$$

ΔX is one count in the 10th place*

This is valid for any X within the range of the function's argument.

Estimates of these upper and lower tolerances may be obtained by using calculated values $\hat{f}(X)$ at plus one count in the tenth place ($+\Delta X$) and minus one count in the tenth place ($-\Delta X$).

This may be stated mathematically:

$$\hat{f}(X - \Delta X) \leq f(X) \leq \hat{f}(X + \Delta X)$$

Graphically the calculated value lies in the regions shown below:

estimated tolerance region	----- $\hat{f}(X + \Delta X)$ (calculated value)	
	$f(X + \Delta X)$ (exact value)	
	tolerance	$\hat{f}(X)$ (calculated value)
	region	$f(X - \Delta X)$ (exact value)
	----- $\hat{f}(X - \Delta X)$ (calculated value)	

Here is an example:

EXAMPLE

```

f(X) = sin X
X = 360 000 003 00
ΔX = 10

X + ΔX = 360 000 003 10
X - ΔX = 360 000 002 90

f(X) = .500 000 000 000

f̂(X + ΔX) = .514 978 232 099
f(X + ΔX) = .515 038 074 910**
f̂(X) = .499 818 609 118
f(X - ΔX) = .484 809 620 246**
f̂(X - ΔX) = .484 504 290 825

```

*It is necessary to specify the tenth rather than the twelfth because of the calculation error.

**The exact values are from National Bureau of Standard's "Handbook of Math Tables."

ERROR SUMMARY

$$\text{Absolute error} = |f(X) - \hat{f}(X)| = .0001814$$

$$\text{Relative error} = \frac{|f(X) - \hat{f}(X)|}{f(X)} = .036\%$$

The following is a summary of maximum calculator function errors. They are expressed as absolute errors unless relative error is noted.

$$\text{Absolute error} = |f(X) - \hat{f}(X)|$$

$$\text{Relative error} = \frac{|f(X) - \hat{f}(X)|}{f(X)}$$

$f(X)$ = exact value

$\hat{f}(X)$ = calculated value

Note:

- a) One visible count is one count in the tenth digit.
- b) "No. of decades of circles" refers to the number of integer circles divided by 10.

- I. ADD(+, -) $\pm 5 \times 10^{-11} \times 10^{\text{(larger addend exponent)}}$
- II. MPY $\pm 1/2$ visible count
- III. DIV $\pm 1/2$ visible count
- IV. SQRT $\pm 1/12$ visible count
- V. COS θ
 - $\theta < 1 \text{ rad}$ $\pm 4 \times 10^{-11}$
 - $1 \text{ rad} \leq \theta < \pi$ $\pm 9 \times 10^{-11}$
 - $\pi \leq \theta < 2\pi$ $\pm 28 \times 10^{-11}$
 - $2\pi \leq \theta$ $\pm 17 \times 10^{-10} \times 10^{\text{(no. of decades of circles)}}$
- VI. SIN θ
 - $\theta < 1 \text{ rad}$ ± 1 visible count
 - $1 \text{ rad} \leq \theta < \pi$ $\pm 12 \times 10^{-11}$
 - $\pi \leq \theta < 2\pi$ $\pm 31 \times 10^{-11}$
 - $2\pi \leq \theta$ $\pm 17 \times 10^{-10} \times 10^{\text{(no. of decades of circles)}}$
- VII. TAN θ
 - $\theta < 1 \text{ rad}$ $\pm 1 1/2$ visible count
 - $1 \text{ rad} \leq \theta < \pi$ $\pm 1/2$ (exp of answer)
 $\pm 2 \times 10^{-10}$ (visible counts)
 - $\pi \leq \theta < 2\pi$ $\pm 1 \times \text{exp of answer}$
 $\pm 3 \times 10^{-10}$ (visible counts)
 - $2\pi \leq \theta$ $\pm 5 \times \text{exp of answer} \times 10^{\text{(no. of decades of visual counts circles)}}$
 $\pm 31 \times 10^{-10} \times 10^{\text{decade of circles}}$
- VIII. POLAR TO RECT same as cos θ and sin θ
- IX. TAN⁻¹(a)
 - radians $\pm 1/10$ visible count
 - degrees $\pm 1/4$ visible count
- X. SIN⁻¹(a) $a < .707$ $\pm 1/2$ visible count
- XI. COS⁻¹(a) $a < .707$ $\pm 1/2$ visible count $\pm 10^{-11}$
- XII. SIN⁻¹(a) $a > .707$ rad $\pm 10^{-10}$
deg $\pm 70 \times 10^{-10}$
- XIII. COS⁻¹(a) $a > .707$ rad $\pm 10^{-10}$
deg $\pm 70 \times 10^{-10}$

- XIV. RECT TO POLAR same as above
- XV. e^θ $\theta < 2.3$ relative error = $\pm 9 \times 10^{-11}$
 $\theta < 23$ relative error = $\pm 3 \times 10^{-10}$
 $\theta < 230$ relative error = $\pm 24 \times 10^{-10}$
- XVI. COSH, SINH $\pm 10^{-10}$ + relative error from e^θ
- XVII. TANH $\pm 10^{-10}$
- XVIII. LN $\pm 7 \times 10^{-11} \pm 1/20$ visible count
- XIX. LOG $\pm 3 \times 10^{-11} \pm 1/8$ visible count
- XX. SINH^{-1} , COSH^{-1} $\pm 24 \times 10^{-11} \pm 1/20$ visible count
- XXI. $\text{TANH}^{-1}(a)$ $\pm 10^{-10} \cosh^2(\tanh^{-1} a)$

CHAPTER III
INSIDE THE CALCULATOR

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Robert Benchley once wrote that although he realized a modern bridge was indeed a very large and complex structure, he nevertheless felt sure he could build one himself, if only he could think of the very first thing to be done. Once over this initial hurdle, he thought, the remaining steps would fall readily into place one after the other until the bridge was complete. It was just his inability to discover the first step that prevented him from becoming a master bridge builder!

You, perhaps, have been trying to discover the first step to be taken in building a knowledge of the 9100A. You have heard that the Calculator can be an extraordinarily powerful tool, that it can make decisions automatically, and solve arithmetic problems at incredible speeds, doing in a matter of seconds an amount of work it would take you days or weeks to accomplish by hand. It turns out that the first step is surprisingly simple. It involves no more than a consideration of the kinds of on-off or up-down patterns you might find represented by a row of simple switches. For the digital computer, and remember the calculator is really a small computer, is basically little more than an automatic device for manipulating just such patterns at high speed according to simple fixed rules.

FIRST STEP

Not that you must understand how the computer works in detail to use it to solve your problem. Not at all! Your main concern is going to be rather the careful formulation of your problem in symbolic terms using pencil and paper. Oh, occasionally you may want to do some fancy programming, and there may be times when interpreting the display in terms of the calculator's internal operation will be helpful. But by and large you will be content simply to "feed" the calculator your problem and its data and have the results returned to you with as little fuss as possible on the CRT, printer or plotter. The fact remains, however, that switch patterns and their manipulation underlie and define everything that the calculator is capable of doing, and it is therefore a good idea to know something about them.

By the way, symbolic procedures for describing calculations are known as algorithms (after the ninth century Arabian arithmetician al-Kuwarizmi, to whom the credit is due for this method of calculating by means of symbol manipulation). Thus you would speak of the binary counting algorithm, or of the familiar grade school decimal counting algorithm, and so forth. All of the fixed rules followed by the calculator as it goes about solving a problem are, in fact, algorithms of one kind or another. The 9100A has both software, the instructions you load, and firmware, the instructions wired into the read-only-memory which generate the keyboard functions.

ALGORITHMS

WHY BINARY?

Binary numbers are much less compact than decimal numbers. They are, however, quite the right sort of numbers to represent in two-state devices as two-position switches. The reason for the binary rather than the more compact decimal system is primarily that two-state devices have been found to be technically and economically sounder than ten-state devices in computer construction.

Two-state devices of various kinds are used extensively. The electronic version of the two-position switch is picturesquely known as a flip-flop. The computer is able to put a flip-flop into either of its two states by means of electrical signals, just as you are able to put a switch into either of its two states by hand. Calculator memory units are composed of a large number of tiny, magnetizable doughnuts misnamed cores, wired together in geometrical arrays and arranged so that each core can be magnetized in one of two ways.

Some specific information on the internal organization of the 9100A's hardware will prove useful. There are innumerable programming "tricks" which can only be used successfully in certain situations. You'll need the information in this chapter to identify those special cases.

NUMBER SYSTEMS

The calculator manipulates instructions and reads data as binary coded decimal numbers. In the binary system, there are only two digits, 0 and 1, which are easily stored and manipulated using bistable devices. The 9100A, for example, converts each key depression into a string of binary digits. For the user, however, binary numbers (such as 1011010011101000) are difficult to read and use, so the bits are grouped in threes for convenient notation in the "octal" numbering system.

Some familiarity with both binary and octal numbering systems, as well as with conversions to and from the decimal system is helpful.

In converting each group of 3 bits to an octal digit, the binary significance of each bit is converted to its absolute value, which is then considered to be absent or present, depending on whether the bit is a "zero" or a "one" respectively.

By various combinations of on and off states, eight digits are possible, 0 through 7. The digits 8 and 9 never appear in the octal numbering system. The Figure on the following page lists all eight binary/octal equivalents, along with some examples of numbers.

<u>Binary</u>		<u>Octal Interpretation</u>		<u>Octal</u>
000	=	0	=	0
001	=	1	=	1
010	=	2	=	2
011	=	2 + 1	=	3
100	=	4	=	4
101	=	4 + 1	=	5
110	=	4 + 2	=	6
111	=	4 + 2 + 1	=	7

5	2	6	0	1	
1 0 1	0 1 0	1 1 0	0 0 0	0 0 1	OCTAL
					BINARY

7	4	3	5	0	
1 1 1	1 0 0	0 1 1	1 0 1	0 0 0	OCTAL
					BINARY

7	7	7	7	7	
1 1 1	1 1 1	1 1 1	1 1 1	1 1 1	OCTAL
					BINARY

EXAMPLES

To avoid confusion when numbers are written in more than one numbering system, a subscripted digit is attached to the number to identify the system used. Thus:

$$1111111111111111_2 = 77777_8 = 32767_{10}$$

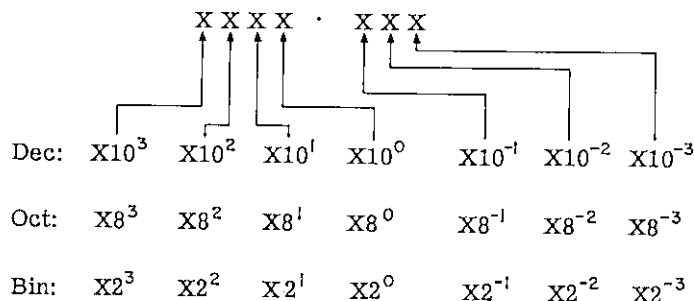
When counting in the octal system, the "carry" to the next more significant column occurs as roll-over from 7_8 to 0_8 occurs. That is, 10_8 follows 7_8 . The counting sequence in octal is:

00000
00001
00002
00003
00004
00005
00006
00007
00010
00011
00012
etc.

OCTAL CARRY

NUMBER CONVERSIONS

Integral and fractional parts of a number are separated by a "decimal point" in the decimal system, an "octal point" in the octal system, and a "binary point" in the binary system. The significance of digit positions in a number in any system increases by positive powers of the system's base when going left from the point, and decreases by negative powers of the system's base when going right from the point. This is shown below.



Significance of Digits in Three Systems

The information in the above Figure provides the basis for converting octal or binary to the decimal system.

Converting octal or binary numbers to the decimal system consists only of performing individual multiplications, each digit times its significance, for all of the digits in the number, and then summing the results. Thus, the octal number 7654.321 has the decimal equivalent of:

$$\begin{array}{rcl}
 7 \times 8^3 & = & 7 \times 512 = 3584. \\
 6 \times 8^2 & = & 6 \times 64 = 384. \\
 5 \times 8^1 & = & 5 \times 8 = 40. \\
 4 \times 8^0 & = & 4 \times 1 = 4. \\
 3 \times 8^{-1} & = & 3 \times 1/8 = .375 \\
 2 \times 8^{-2} & = & 2 \times 1/64 = .03125 \\
 1 \times 8^{-3} & = & 1 \times 1/512 = .001953125 \\
 \hline
 & & 4012.408203125
 \end{array}$$

Octal to
Decimal

Decimal
to Octal

Integral and fractional parts of a decimal number require separate operations when converting to the binary or octal system. Because of this added complexity and the ease of octal/binary conversion, one should limit decimal conversions to octal only, and then to construct the binary equivalent if necessary from the octal number.

Basically, the procedure for the integral part of the number is first to divide the new base (8, if converting to octal) into this part of the

number, stopping at the decimal point. The resulting number is a whole number and a fractional remainder (e.g., $32767 \div 8 = 4095$ plus a remainder of 7 eighths). The remainder (7) becomes the least significant integer of the new number being constructed (i.e., immediately to the left of the octal point). The whole portion (4095) is again divided by the base (8), and the process is continued until the whole portion is reduced to zero.

$$\begin{array}{rcl}
 32767 \div 8 = & 4095 & + \\
 4095 \div 8 = & 511 & + \\
 511 \div 8 = & 63 & + \\
 63 \div 8 = & 7 & + \\
 7 \div 8 = & 0 & +
 \end{array}
 \begin{array}{c}
 7 \\
 7 \\
 7 \\
 7 \\
 7
 \end{array}
 \begin{array}{c}
 \downarrow \\
 \downarrow \\
 \downarrow \\
 \downarrow \\
 \downarrow
 \end{array}
 \begin{array}{c}
 7 \\
 7 \\
 7 \\
 7 \\
 7
 \end{array}
 .8$$

To convert the fractional part of a decimal number to octal, multiply by the base and use the whole portion of the resulting number as the first digit to the right of the octal point. Continue by multiplying the fractional part of the same resulting number by the base again, to as many places of accuracy as desired. Thus 0.135 decimal is approximately:

$$\begin{array}{rcl}
 .135 & .08 & .64 & .12 & .96 \\
 \times 8 & \times 8 & \times 8 & \times 8 & \times 8 \\
 \hline
 1.080 & 0.64 & 5.12 & 0.96 & 7.68
 \end{array}$$

Digital data can also be coded in Binary Coded Decimal. Four binary bits are required to code each decimal digit. Since this is a code rather than a number system, there are no carries. Many BCD weightings could be devised. The most common, and the one used in the 9100A is 1-2-4-8.

BCD WEIGHT	8421	8421	8421
BCD CODE	0101	0110	1000
DECIMAL DIGIT	5	6	8

BINARY
CODED
DECIMAL

For single digits less than eight, the BCD code corresponds to octal.

PROGRAM STORAGE

Your program is stored as a precisely defined set of Calculator operations. These instructions are stored in one or more of the Calculator's 14 program registers. Each register in turn has 14 sublocations for storing instructions. These are stored as two digit octal numbers; i. e., the instruction CLEAR has the code 20. Upon encountering this code in a memory subregister, the Calculator will automatically clear the X, Y, Z, e and f registers, plus the FLAG.

The 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, and d registers can be used for either program or data storage. If all are used for program storage, 196 program steps are available. The X, Y, and Z registers are display registers and cannot be used for program storage; the e and f registers are normally used for data storage; like X, Y, and Z, they cannot be recorded on a magnetic program card.

STARTING ADDRESS

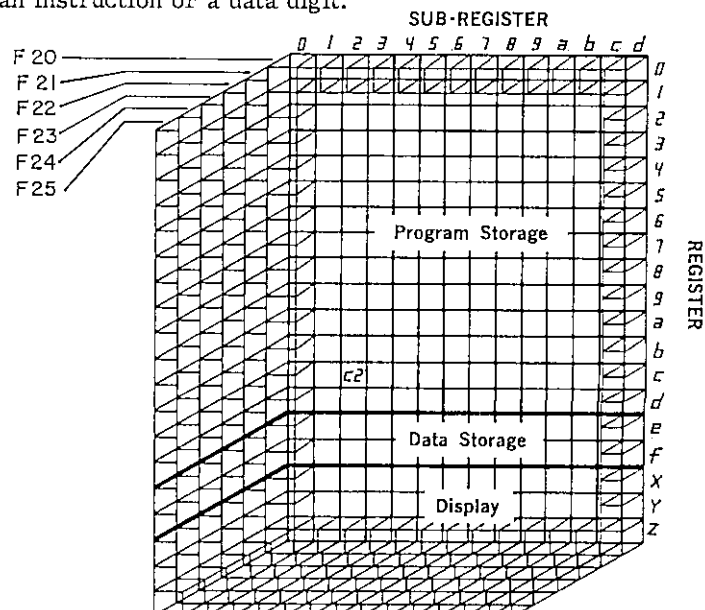
A program is stored so that the register with the starting address is filled with program steps before the next sequential register is entered with program steps. With the first program step in location 00, program storage will be used in the following manner: 00, 01, ... 09, 0a, 0b, 0c, 0d, 10, ... 19, 1a, 1b, 1c, 1d, 20 ... up to dc, dd, the address at which the program counter resets at 00.

SPACE TRADE-OFF

Programs that require more data storage than the e and f registers must use a register that could otherwise be used for program storage. A complete register is required for storing a twelve-digit number and the two-digit exponent; therefore, the maximum program size of 196 steps is reduced by 14 steps for each program register used for data storage.

The core memory used for program storage consists of a 14x14x6 array, figure below. The registers are designated 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, and d, as are the sublocations in each register. Memory address c2 indicates the c register and sub-register location 2 in the c register. Each memory address has six switches for coding an instruction or a data digit.

CORE MAP



The calculator writes instruction codes in the sub-register by setting each switch either on or off. Instructions are read by interpreting the switch setting. Each switch position has an associated number or weighting. If the switch is on, that number is read and summed. Otherwise, it is not. The six switch positions and associated weights are:

WEIGHT	40	20	10	4	2	1
SWITCH NO.	F25	F24	F23	F22	F21	F20

Let "1" represent an "on" switch and "0" represent an "off" switch. Then 010000 represents the code 20 which is the instruction to CLEAR. Code 35 is the instruction to DIVIDE - 011101.

INSTRUCTION FORMAT

The Calculator uses these same switches for storing data, which may be placed in the memory registers instead of instructions. However, the weight or meaning of each switch is different if data is stored in a register.

WEIGHT	Blanking	Sign + or -	8	4	2	1
SWITCH NO.	F25	F24	F23	F22	F21	F20

Switch F25 is used for determining whether a zero is significant and should be displayed, or is not significant and should be blanked. If the blanking switch is on (F25=1), the zero is not displayed. So, the digit 0, if significant and displayed, is represented as 000000. The digit 6 is represented as 000110.

DATA FORMAT

Here is the way each of these subregisters is arranged as displayed in floating point form.

DIGIT:	D_s	D_9	D_8	D_7	D_6	D_5	D_4	D_3	D_2	D_1	D_0	G_1	G_0	E_s	E_1	E_0
SUBREGISTER		9	8	7	6	5	4	3	2	1	0	d	c	b	a	
LOCATION:																
DATA:		3	.	1	4	1	5	9	2	6	5	3	6	0	0	0

DISPLAY

D_s is the sign of the number and is set by F24 of the switch set D_9 . D_9 is the location of the most significant digit. E_s is the sign of the exponent and is set by F24 of the switch set E_1 . G_1 and G_0 are guard digits (for rounding) and are not displayed.

You will probably notice that when data is stored the digits are not stored left to right, but rather with the least significant digit in the first memory sub-register. Try the following:

- Place Calculator in RUN mode
- GO TO (a) (0)
- Switch to PROGRAM mode
- Press successively the digits keys
1, 2, 3, 4, 5, 6, 7, 8, 9, 8, 7, 6, 5, 4

GUARD DIGIT DISPLAY

- Return to RUN mode
- Press a

As you can see, the most significant digit is D₉, and it appears just to the right of center on the CRT. What are the values of the two guard digits?

You can view the guard digits by subtracting the first two digits in the display thus creating room for the guard digits.

- Press UP
- Enter digits 8, 9
- Press ENTER EXPONENT 67
- Press —

The two digits which were previously guard digits have shifted left two places; they are now displayed as the two least significant digits in the Y register. Is the display what you expected?

You should also try the reverse operation:

- Set RUN mode
- Press π
- Press X to a
- Press GO TO (a) (0)
- Set Program mode
- Read the digits out using STEP PROG

DATA AS INSTRUCTIONS

It is possible to interpret and display data digits as instruction codes. When single stepping the Calculator in the PROGRAM mode, the display subroutine always decodes all six bits in each character location as instructions--it has no way of knowing when data rather than an instruction is stored.

With these decimal numbers stored, the bit pattern and, therefore, the display would be as follows:

DECIMAL NUMBER	F25 40	F24 20	F23 10	F22 4	F21 2	F20 1	DISPLAY CODE
9	0	0	1	0	0	1	= 11
8	0	0	1	0	0	0	= 10
4	0	0	0	1	0	0	= 4
-3	0	1	0	0	1	1	= 23
blank digit	1	0	0	0	0	0	= 40

So 40 will show up any time a blank digit is stored. Occasionally, a customer debugging a program will encounter this and get confused. It usually means that they have stored data in a register which should have been reserved for program storage.

It is also possible to interpret and display instruction codes as data. Try entering instructions from the storage section of the keyboard in a - 0 through a - d and then switch to RUN and recall a. What are you looking at?

INSTRUCTIONS
AS DATA

CHAPTER IV
P R O G R A M M I N G

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There are many steps involved in problem solving. Only a few of these are accomplished by the Calculator itself. Most others require the work of problem solvers. It will be instructive to outline the steps involved in setting up a typical problem for computer solution.

PROBLEM SOLVING

One must identify and define the problem to be solved. A general approach should be decided upon. In some applications this is simple; in others it is quite difficult and time consuming. In any case this process requires full knowledge of the problem and the goals to be met.

Identification

There are usually several ways to mathematically describe a problem; one must be chosen or a new one developed if no standard exists. This clearly requires knowledge of the relevant fields of mathematics.

Description

Since the Calculator can only do arithmetic and make simple quantitative decisions, it may be necessary to modify the mathematical formation of the problem so that it is translatable to Calculator program operations.

Analysis

The numerical procedure must be stated as a precisely defined set of Calculator operations. This is usually done in two steps. The first is a graphical representation of the procedure called a flow-chart. The procedure is then written as a set of Calculator operations.

Programming

In a long program there are so many chances to make a mistake that the program will not usually work when first tried. It will be shown how the Calculator is used to verify and edit the program.

Verification

Now the program can be combined with problem data and executed. The step typically takes only a few seconds but may take several hours.

Execution

Often the results displayed by the Calculator do not constitute the final answer to the problem. The user must interpret the results to see what they mean in terms of the goals that the proposed system must meet. It may be necessary to repeat some of the preceding steps before the problem is really solved.

Interpretation

The role of the computer (Calculator) in problem solving is succinctly defined by Danial D. McCracken.

Summary

"First, the computer does not, by itself, solve problems; it only follows carefully defined computational procedures. Second, a computer does not relieve the user of the responsibility of planning the work carefully; in fact, the computer demands much more careful planning. The computer is faster and more accurate than a human, but it cannot decide how to proceed or what to do with the results. Third, a computer does not in any way reduce the need for a full and detailed understanding of the problem area or for a thorough knowledge of the related mathematics."

FLOWCHARTING

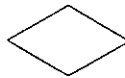
An important tool of programming is the flow-chart, or block diagram, which allows the programmer to plan the sequence of operations in a program before coding it. In any problem that is even moderately complex the interrelationships within it are difficult to keep clearly in mind without some visual representation. A flow-chart provides this visual assistance. It also greatly facilitates communication between the programmers and is a valuable part of the documentation of a program.

A flowchart is made up of a set of boxes, the shapes of which indicate the nature of the operations described in the boxes, along with connecting lines and arrows that show the "flow of control" between the various operations.

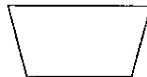
The most common notation is shown here:



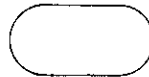
A rectangle indicates any processing operation except a decision.



A diamond indicates a decision. The lines leaving the box are labeled with the decision results that cause each path to be followed.



A trapezoid indicates an input or output operation.



An oval indicates the beginning or ending of the program.

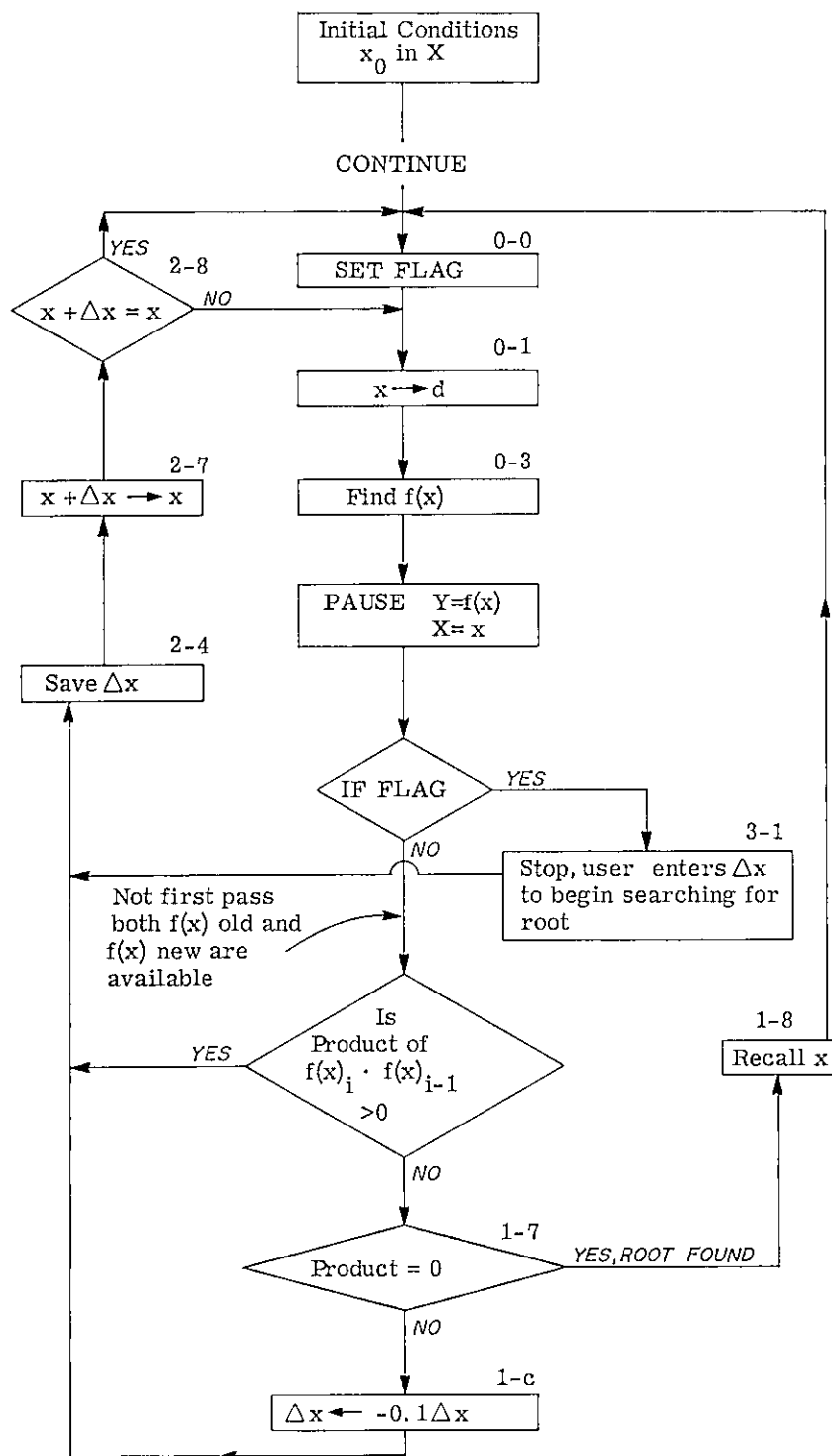


A small circle indicates a connection between two points in a flowchart, where a connecting line would be too clumsy.



Arrows indicate the direction of flow through the flowchart; every line should have an arrow on it.

Here is an example from the Program Library (09100-70014). The program finds the real roots of $f(x)$.

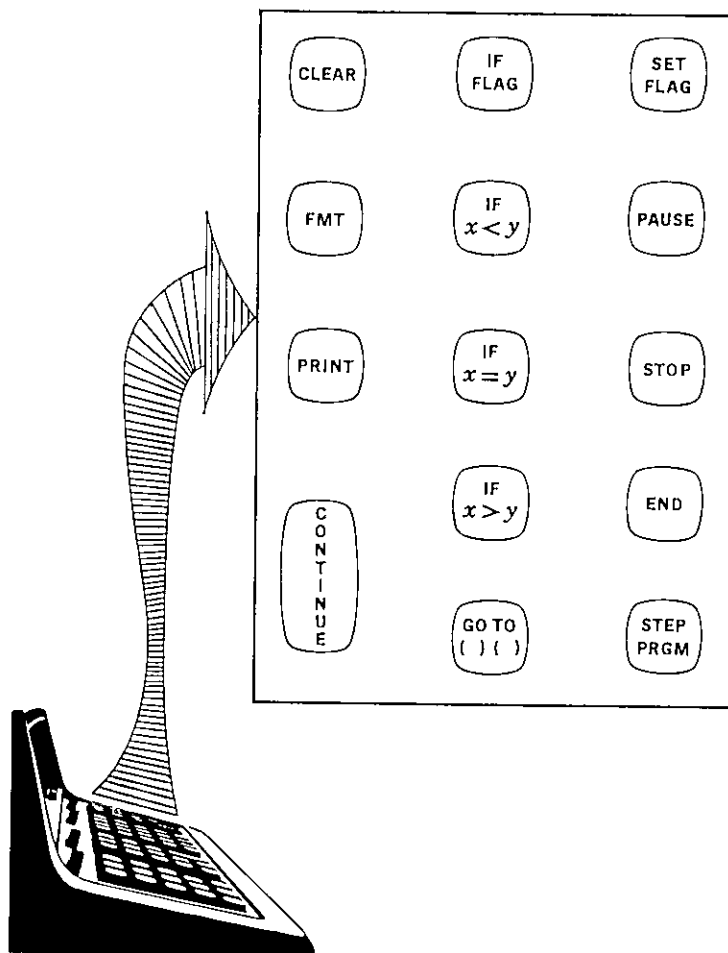


PROGRAMMING

Once your flowchart is written it is necessary to transform it to instructions which the calculator can interpret. However, it is not necessary to learn a program language for the Calculator. Keyboard operations are the program instructions. Each program step is one keyboard instruction. So programming the calculator is as simple as writing your equation in sequential steps.

Branching and looping instructions can be easily inserted in your program by using the qualifying IF keys - which give you capabilities usually found only on large computers. Changes in your program are simple to make, too. You may address and correct an individual step. You do not have to re-enter the entire program.

Once you have your program in the calculator, you can make a permanent record of the program on a wallet size magnetic programming card. Now your program may be re-entered directly from the magnetic card at any time.



Selects the mode of operation.

RUN: used to perform calculations, address the program counter and execute a program.

PROGRAM: used to enter and verify a program.

PROGRAM RUN

Causes an unconditional branch to the program address entered by the next two keystrokes and by the next two program steps. Each key-stroke is one program step.

GO TO
() ()

CORRECT

Step	Key
29	GO TO () ()
2a	5
2b	c

INCORRECT

Step	Key
29	GO TO () ()
2a	5c
2b	

Starts program execution at the present address. If the program has halted execution at a STOP instruction, pressing CONTINUE will start program execution with the next program step. If the program has halted execution at an END instruction, pressing CONTINUE will start program execution at program step 00. CONTINUE can also be used as a "no operation" program step. See the program for N!, page 51.

CONTINUE

Halts the program execution and causes a return to the display mode of operation. The STOP instruction provides an unconditional halt of program execution for entry of data. Pressing CONTINUE begins program execution with the next program step. Pressing STOP during execution of the program will cause the program to stop execution at the present program step and return to the display mode. Pressing CONTINUE will begin program execution with the next program step.

STOP

Use the STOP instruction to enter data. With the program at a STOP instruction any keyboard operation can be manually executed, provided that no data is stored in a register containing program steps. Storing data in a register containing program steps will destroy the program steps in that register.

END must be the last instruction in a program. Halts the program execution and causes a return to the display mode. The instruction resets the program counter to 00 so that pressing CONTINUE begins program execution at step 00.

END

RECORD: All of core memory is recorded (re-cycling to 00 from dd) until the 9100A "runs out" of card.

ENTER: Program is read to first END instruction which inhibits the reading of any more program steps. Without an END, the 9100A reads until it "runs out" of card; if this happens in the middle of the program, a program step will be incorrectly entered. The incorrect step: (1) will vary over about six step range, and (2) will vary each time the program is entered.

Causes a brief 1/8 second display before continuing with execution of the program. If the PAUSE key is held depressed during execution of a program a stop will occur at the next PAUSE instruction in the program. Pressing CONTINUE will begin execution with the next program step. Use successive PAUSE instructions for a longer display.

PAUSE

STEP
PRGM

PROGRAM

RUN

PRINT

FMT

ENTERING A PROGRAM

This instruction single-steps the program. The display will depend upon the mode of operation: PROGRAM or RUN. This is the only instruction that cannot be a program step.

Single-steps the program, displaying in X the program step address and the code of the instruction in that program step.

12-----54

Program step 12 contains the SET FLAG instruction; its code is 54.

Single-steps through execution of the program displaying the contents of the X, Y, and Z registers after each step. Enter data at the proper program step to check operation of the program.

If a CONTINUE instruction is encountered, the Calculator will begin execution of the program. To verify a program with RUN selected, replace each CONTINUE instruction with either a PAUSE or STOP instruction.

Prints the contents of X, or X and Y or X, Y and Z as selected on the printer. If no printer is being used, the PRINT instruction will act as a STOP instruction.

The FORMAT instruction controls the accessory output device. If no accessory output device is being used, the FORMAT instruction will act as a STOP instruction.

FROM THE KEYBOARD

1. SET: RUN
2. PRESS: GO TO () ()
Starting Address
3. SET: PROG
4. Press keys in program sequence.
5. SET: RUN
6. PRESS: GO TO () ()
Starting address
The program will execute the instruction at the starting address when CONTINUE is pressed.
7. Enter data and press CONTINUE as necessary for program execution.

FROM A MAGNETIC CARD

1. SET: RUN
2. PRESS: GO TO () ()
Starting Address
3. Insert card, PRESS: ENTER
4. PRESS: GO TO () ()
Starting Address
5. Enter data and press CONTINUE as necessary for program execution

FROM THE KEYBOARD (con't)

When entering a program, the address for the instruction entered and code of that instruction are displayed in the X register.

A.	00
A.	00
Z.C - - - -	33

Step 2-c contains the + (add) instruction.

See "CORRECTING A PROGRAM" (page 46) if an incorrect instruction is entered in the program.

Any address can be used for the first program step. However, a beginning address of 0 - 0 is generally used because of the convenience provided by the END instruction. Core storage is usually used beginning at the top (location 0-0) for program storage and beginning at the bottom (register f) for data storage.

1. SET: ☐ RUN
2. PRESS: GO TO () () (Starting address)
3. INSERT CARD, PRESS: RECORD

Two programs can be recorded on each magnetic card. The programs are designated A and B. To read or record program A, insert the card with the A arrow pointing down and the printed side of the card facing the user. To read or record program B, insert the card in the same manner with the B arrow pointing down.

To permanently save a program record as program A, cut off the corner of the card along the tip of the A arrow.

After a program is entered in storage it can be verified by selecting either the PROGRAM or RUN mode of operation.

1. SET: ☐ RUN
2. PRESS: GO TO () () (Starting Address)
3. SET: ☐ PROG ☐
4. Each Step PRGM keystroke will increment the program counter. The X register will display the program step address and the code of the instruction in that location.

FROM A MAGNETIC CARD (con't)

When entering or recording a program;

(1) Hold the ENTER or RECORD key pressed until the card is partially ejected from the card reader.

(2) The contents of the X register are destroyed and the contents of Y and Z are unchanged.

RECORDING A PROGRAM

VERIFYING A PROGRAM

Program
Mode

Run Mode

1. SET: RUN
2. PRESS: GO TO () () (Starting Address)
3. Each STEP PRGM keystroke will single-step the program, displaying the contents of the X, Y, and Z registers. Enter data at the correct program step to check operation and solution of the program. While editing, replace every CONTINUE instruction with a PAUSE or STOP instruction. Encountering a CONTINUE instruction while editing in the RUN mode of operation will cause the Calculator to begin automatic execution of the program.

The key codes are given on Page 154 and on the pull-out instruction card.

CORRECTING A PROGRAM STEP

1. SET: RUN
2. PRESS: GO TO () () (Step to be changed)
3. SET: PROG
4. ENTER CORRECT INSTRUCTION
5. a. TO CONTINUE EDITING:
PRESS: STEP PRGM
b. TO EXECUTE PROGRAM
SET: RUN
PRESS: GO TO () () (Starting Address)

PROGRAMS AND EXERCISES

This section contains sample programs to illustrate use of the arithmetic, function, and storage keys. It will be helpful to enter the sample programs and, using the STEP PROGRAM key in the run mode, execute each instruction while watching the display.

Write a program to solve $\frac{A \times B}{A + B}$ for any A and B.

STEP	KEY	x	y	z	e	f	
0-0	CLEAR	0	0	0	0	0	CLEAR DISPLAY & e & f
0-1	STOP	A	B				ENTER B AND A
0-2	$x \rightarrow ()$						
0-3	E				A		STORE A AND B
0-4	$y \rightarrow ()$						
0-5	f	A	B	0		B	
0-6	$\times$		A $\times$ B				
0-7	$\uparrow$		A	A $\times$ B			MOVE A $\times$ B TO Z TO
0-8	f	B					CALCULATE A + B
0-9	$+$		A + B				
0-a	ROLL $\downarrow$	A + B	A $\times$ B	B			POSITION A + B AND A $\times$ B
0-b	$\div$		$\frac{A \times B}{A + B}$				TO CALCULATE QUOTIENT
0-c	ROLL $\uparrow$	B	A + B	$\frac{A \times B}{A + B}$			
0-d	$x \leftrightarrow y$		B				
1-0	E	A					
1-1	END	A	B	$\frac{A \times B}{A + B}$			FINAL DISPLAY

Write a program to count by N.

0-0	CLEAR				
0-1	STOP	n			ENTER n.
0-2	↑	n	n+n		DISPLAY; x CONTAINS n y CONTAINS TOTAL
0-3	+				
0-4	PAUSE				
0-5	GO TO () ()				
0-6	0				} BRANCH TO CONTINUE SUMMATION
0-7	3				
0-8	END				
STEP	KEY	x	y	z	

The program $\frac{A \times B}{A + B}$ may be written with thirteen steps. The first instruction should be CLEAR. The last instruction should be END displaying A, B, $\frac{A \times B}{A + B}$ in the X, Y, and Z registers respectively. Notice it was not necessary to use any of the storage registers.

EXERCISES

Write a program to find the value of: $x^2 + bx + c$ for real values of x, b, c. An efficient program may be written in 9 steps. At END display:

Z	0
Y	0
X	$x^2 + bx + c$

Begin with CLEAR and STOP for the user to enter his values of x, b, and c.

Write a program to find the roots of the equation $ax^2 + bx + c$. There are several ways of expressing the roots. Try different algebraic expressions and flowchart the program to see which is most efficient. Be sure to account for complex and double roots. Does your program use less than four registers?

Case 1: a = 2
b = 9
c = 4

Answer:
-0.5, -4.0

Case 2: a = 1
b = -2
c = 1

Answer:
1, 1

Case 3: a = 1
b = 3
c = 6.25

Answer:
-1.5 + j2
-1.5 - j2

BRANCHING LOOPING AND CONTROL KEYS

IF
 $x < y$

IF
 $x = y$

IF
 $x > y$

IF
FLAG

SET
FLAG

These instructions cause a conditional branch to one of two program locations dependent upon the contents of the X and Y registers.

IF $X < Y$
IF $X = Y$
IF $X > Y$

If the condition tested is met, the program will either branch to the address stored in the next two program steps or, if no address is given, execute the instructions in the next two program steps, and continue with the third program step following the conditional branch instruction. If the condition tested is not met, the program branches to and executes the third program step following the conditional branch instruction.

If the contents of $X = 6$, and the contents of $Y = 5$, the condition (IF $X > Y$) is met. The program will branch to step 08, the address given in steps 03 and 04, and enter 1. in the X and Y registers. If the contents of $X = 2$, and the contents of $Y = 5$, the condition is not met. The program will branch to step 05 and enter 2. in the X and Y registers.

STEP	KEY
0-0	CLEAR
0-1	STOP
0-2	IF $x > y$
0-3	0
0-4	8
0-5	2

STEP	KEY
0-6	↑
0-7	STOP
0-8	1
0-9	↑
0-a	END

If the contents of X and Y are equal, the condition (IF $X = Y$) is met. The program will execute steps 03 and 04 and continue with step 05, calculating $x + y$ and entering the answer in X. If the contents of X and Y are not equal the program will branch to step 05, calculate the product of $X \times Y$ and enter it in X.

STEP	KEY
0-0	CLEAR
0-1	STOP
0-2	IF $x = y$
0-3	TO POLAR

STEP	KEY
0-4	↑
0-5	X
0-6	↓
0-7	END

This instruction causes a conditional branch to one of two program steps depending upon whether or not the FLAG was set, either manually or by a program step, with the SET FLAG instruction. The condition is met if the FLAG was set. Branching after the IF FLAG instruction is the same as for the conditional branch instructions given above. The instruction also clears the FLAG.

Set the FLAG condition which is tested by the conditional branch instruction IF FLAG. This instruction and the IF FLAG instruction allow the operator to control program execution by either manual entry or program step entry of the SET FLAG instruction.

If the FLAG is set, multiply the contents of X and D and branch to the beginning of the program. If the FLAG is not set, multiply the contents of Y and d and halt execution of the program.

EXAMPLE

SOLUTION

STEP	KEY
0-0	CLEAR
0-1	STOP
0-2	IF FLAG
0-3	SET FLAG
0-4	$yz()$
0-5	d
0-6	$\times$
0-7	IF FLAG
0-8	0
0-9	1
0-a	END

IF THE FLAG IS SET

0-2 Continue with step 03 and clear the FLAG
 0-3 Set the FLAG; we'll need it again
 0-4 Exchange the contents of Y
 0-5 With the contents of d
 0-6 $X \times Y = X \times d$
 0-7 Branch to step 0-1 (address given in steps 08 and 09) and clear the FLAG.

IF THE FLAG IS NOT SET

0-2 Branch to step 0-5
 0-5 Recall d to the X register
 0-6 $X \times Y = d \times Y$
 0-7 Branch to step 0a and END.

The only sequence of instructions after a conditional branch instruction which will not give the expected results is a numeric or alphabetic instruction which will not give the expected results is a numeric or alphabetic instruction followed by a non-arithmetic or non-alphabetic instruction. The program will branch to the address (X) (Y) given by the numeric or alphabetic instruction (X) and the second digit (Y) of the third instruction following the conditional branch instruction. The program will also execute the instruction in the second program step following the conditional branch instruction.

STEP	KEY
0-4	IF $x < y$
0-5	2
0-6	ROLL $\downarrow$
0-7	$\div$

EXAMPLE

If the contents of $X = 3$, and the contents of $Y = 5$, the condition (IF $X < Y$) is met; the program will continue to step 05. The ROLL instruction will be executed and the program will then branch to step 27. This is determined by the 2 in step 05 and the second digit of the coded instruction in step 06, rather than entering 2 in the X register and performing the ROLL.

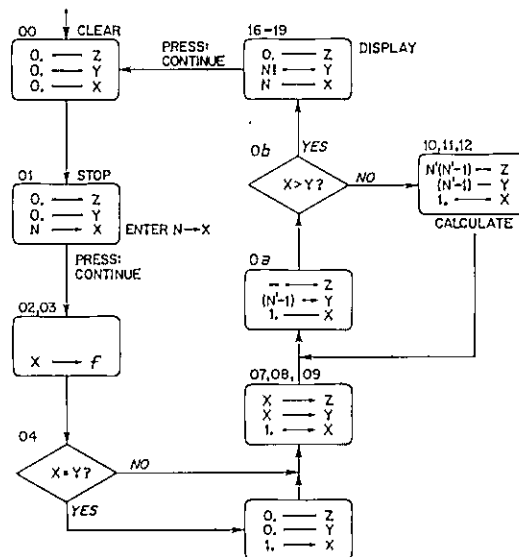
EXAMPLE

CALCULATE N FACTORIAL ($N!$)

$$N! = N(N-1)(N-2) \dots (N-N+2)(1)$$

$$6! = 6(5)(4)(3)(2)(1) = 720$$

$$0! = 1$$



The IF $X=Y$ is met at step 04 if $N=0$; set $0! = 1$. $N!$ is the partial product calculation. Each IF $X>Y$ at step 05 not met develops next partial product. After 1st pass, $N! = N(N-1)$. After 2nd pass, $N! = N(N-1)(N-2)$

CALCULATE N FACTORIAL (N!)

STEP	KEY	x	y	z	f	e
0-0	CLEAR	0	0	0	0	0
0-1	STOP	N	ENTER N → X			
0-2	$x \rightarrow ()$					
0-3	f				N	
0-4	IF $x=y$					
0-5	CONTINUE					
0-6	1	1				
0-7	$\uparrow$	N	N			
0-8	$\uparrow$	N	N	N		
0-9	1	1				
0-a	$-$	1	N-1	N		
0-b	IF $x > y$					
0-c	1					
0-d	6					
1-0	ROLL $\downarrow$	N-1	N	1		
1-1	$\times$		N(N-1)			
1-2	ROLL $\uparrow$	1	N-1	N(N-1)		
1-3	GO TO () ()					
1-4	0					
1-5	9					
1-6	0	0	N-1	N!		
1-7	ROLL $\downarrow$		N!	0		
1-8	f	N	N!	0		
1-9	END	N	N!	0	DISPLAY	

A blank register indicates the contents of the register are unchanged from the previous step.

This is a good example of using the ACC+ key and a counting routine.

EXAMPLE

Calculate $\frac{\sum X_i}{N}$, $\frac{\sum X_i^2}{N}$ where X_i are i data points, and n is the number of data points.

EXERCISE PROBLEMS

STEP	KEY	x	y	z	f	e	d	
0-0	CLEAR							
1	$x \rightarrow ()$							STORE COUNTER (N)
2	d							
3	STOP	X_1						
4	$\uparrow$							
5	$\times$	X_1	X_1^2		ΣX_1	ΣX_1^2		
6	ACC $+$							
7	d	N						INCREMENT COUNTER
8	$\uparrow$		N					
9	1	1						
a	$+$		N+1					
b	$y \rightarrow ()$							
c	d						N+1	
d	IF FLAG							IS THERE MORE DATA?
1-0	1							
1	5							
2	GO TO () ()							BRANCH FOR NEXT DATA ENTRY
3	0							
4	3							
5	$\uparrow$			N				COMPLETE CALCULATION AND DISPLAY RESULT
6	RCL	ΣX_1	ΣX_1^2					
7	ROLL $\uparrow$	N	ΣX_1	ΣX_1^2				
8	$\div$		$\Sigma X_1 / N$					
9	$x \leftrightarrow y$	$\Sigma X_1 / N$	N					
a	ROLL $\downarrow$	N	ΣX_1^2	$\Sigma X_1 / N$				
b	$\div$		$\Sigma X_1^2 / N$					
c	END	N	$\Sigma X_1^2 / N$	$\Sigma X_1 / N$				

Write a program to find the sum of the odd numbers from 1 to N. Your program should stop for the operator to enter his value of N. An efficient program has 23 steps. Display N in X and the sum in Y.

Write a program to take repeated square roots until N is ≈ 1.000000000 count the number of square roots taken. Your program should stop for the operator to enter his value of N.

Write a program to find $\sin X$ accurate to 10^{-9} . Use

$$\sin X = X - \frac{X^3}{3!} + \frac{X^5}{5!} \dots$$

In this program it is important to consider execution time as well as minimizing the program steps. Place a PAUSE after calculating each term to display it. Count the number of terms in the series. Check your degree-radian switch.

These are the first things to check for if a customer calls and says that his diagnostic program works, but that his own program doesn't.

1. Each keystroke is ONE program step.
2. If you add or remove program steps it may be necessary to change the addresses of your branching instructions.
3. Do not enter the decimal point in the program address--even though the address is displayed with a decimal point on the CRT.
4. The contents of the address xx are not displayed when in the "RUN" mode. The Calculator is commanded to "GO TO" xx and is then switched to the program mode. It is necessary to press STEP PROGRAM once in order to display the address and contents. Or you may also press the first key in your program and it will be correctly stored in location xx.
5. Be sure the DEGREE-RADIAN switch is properly positioned before executing your program.
6. Encountering a CONTINUE instruction while stepping a program with "RUN" selected automatically continues execution of all instructions until a STOP or END instruction is encountered. For verifying a program with "RUN" selected, replace each CONTINUE instruction with a PAUSE, STOP.
7. When programming, do not enter the instruction code number, simply depress the desired key and the proper code is automatically entered.
8. The Calculator records instructions from memory onto a magnetic card until an END statement is encountered. Be sure the last instruction you want recorded is an END instruction, and there are no END statements in the middle of your program.
9. When entering a program with the magnetic card reader, be sure the mode switch is in RUN, rather than PROGRAM.
10. If a conditional statement tests true and one wants to branch, it is not necessary to use a "GO TO" statement after the second instruction following the conditional statement.
11. If you wish to branch to a new location on the false condition of a conditional instruction, you must use GO TO in the third step after the qualifier.
12. If program steps stored in a particular register are recalled to the X or Y registers, the program step in the a or b subregister may be destroyed.
13. If you see a string of octal code, 40, you're looking at a register which has had data stored on top of your instructions.

While writing programs for the Calculator, users have discovered and documented special programming techniques which may help you to make your program more efficient.

COMMONLY CONFUSING PROGRAMMING OPERATIONS

SPECIAL PROGRAMMING TECHNIQUES

ENTERING A CONSTANT

During program execution a constant can be entered in the X register by program step entry rather than using a complete register to store the constant and recalling it when needed.

2 3 . 6

After program step 26 is executed, 2.13 will be in the X register. In cases where a constant is entered and used many times it may be more efficient to store it rather than re-enter each time it is required.

SUM OF THE SQUARES

Problem: Calculate $\sqrt{P^2 + Q^2}$

STEP	KEY	STEP	KEY
0-0	ENTER P	0-5	X
0-1	↑	0-6	↓
0-2	X	0-7	+
0-3	ENTER Q	0-8	↓
0-4	↑	0-9	$\sqrt{x}$

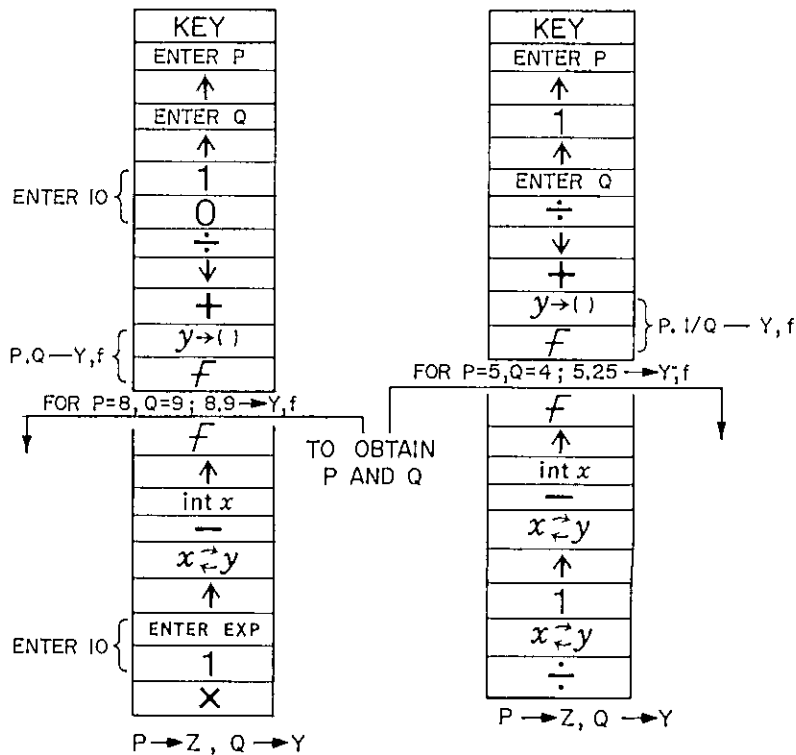
$\sqrt{P^2 + Q^2} \rightarrow X$

STEP	KEY
0-0	ENTER P
0-1	↑
0-2	ENTER Q
0-3	TO POLAR

$\sqrt{P^2 + Q^2} \rightarrow X$

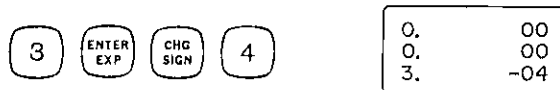
SPLITTING A REGISTER

Problem: Save a storage register by storing two integers P and Q in one register.



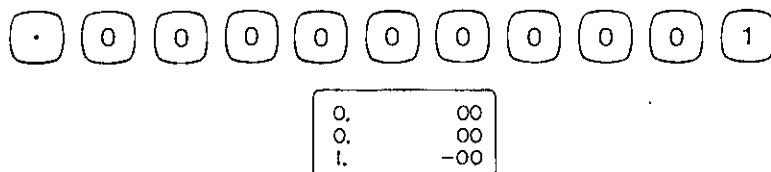
Pressing ENTER EXP causes the following CHG SIGN and digit entries to be entered as the exponent of X. After ENTER EXP is pressed, pressing the decimal point key will return the control of the CHG SIGN and digit keys to the number in X.

NUMBER
ENTRY



Any number with an absolute value of less than 1×10^{-9} must be entered using the ENTER EXP key. It cannot be entered in decimal format.

For example, 1×10^{-10}



The e and f registers can be used for program steps. However, each must be addressed with a GO TO (e) (0) or a GO TO (f) (0) to enter program steps. e0 is displayed as -.0 and f0 is displayed as 10. The step following ed is a0 and following fd is c0. Branching to any other address requires direct branching with the GO TO () () instruction. The e and f registers cannot be recorded on the magnetic card.

Frequently, it becomes necessary to store a group of constants in just one register. The following routine shows how each constant is stored in one register and later isolated for recall.

PACKING A
REGISTER

Twelve digits per register may be used for packing constants. (Even though only ten digits per register are displayed twelve digits are carried in the register). Assume the following constants are to be stored in one register: 18, 63, 15, 9, 79, 80. This can be accomplished by storing them as one compound number 186315097980. Note that a zero has been added to precede the 9 so that each constant has the same number of digits. The following program stores all the constants in register f and extracts one of them; in this case 15. The program is easily modified to extract any constant from any group of 1-6 digit constants.

STEP	KEY	x	y	z
0-0	CLEAR			
1	STOP	186315097980	0	0
2	$x \rightarrow ()$			
3	f			
4	$\uparrow$		186315097980	
5	ENTER EXP	1,000		
6	8	1.0×10^8		
7	$\div$		1863.15097980	
8	$\downarrow$	1863.15097980	0	
9	$\uparrow$		1863.15097980	
a	int x	1863.		
b	$-$		.15097980	
c	ENTER EXP	1.0		
d	2	1.0×10^2		
1-0	X		15.097980	
1	$\downarrow$	15.097980	0	
2	int x	15		
3	END	15	0	0

ENTER COMPOUND
NUMBER IN X

STORE IN f

MOVE DECIMAL TO
LEFT OF DESIRED
CONSTANT

ERASE CONSTANTS
TO LEFT OF
DESIRED ONE

ERASE CONSTANTS
TO RIGHT OF
DESIRED ONE

PROGRAM STEPS & DATA IN SAME REGISTER

"Can you store program steps and a constant in one storage register?" In a manner of speaking, YES. One useful application is to use the data half of this split register as a counter since we don't need all 12 digits of the number (we never count to 10^{12}). However, there are two restrictions. First, let's see how an add operation with a carry from D_9 is performed.

Assume:

$$\begin{array}{rcccccccccccc}
 & D_9 & D_8 & D_7 & D_6 & D_5 & D_4 & D_3 & D_2 & D_1 & D_0 & G_1 & G_0 & E_1 & E_0 \\
 Y = & 9 & 8 & 7 & 6 & 5 & 4 & 3 & & & & & & 0 & 0 \\
 X = & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & 0 & 0 \\
 \text{and we push the } \boxed{+} \text{ key.}
 \end{array}$$

G_0, G_1, D_0, D_1, D_2 are zero in X and Y. An addition will result in 987 654 in the $D_8 - D_3$ positions of the Y register. When the most significant digits (D_9) are added a carry bit is generated. D_9 can't be 10 so the result in Y must be normalized; in this case, shifted to the right (as you view the CRT) and the exponent adjusted accordingly (incremented by one). After the shift of the answer Y contains:

$$\begin{array}{rcccccccccccc}
 D_9 & D_8 & D_7 & D_6 & D_5 & D_4 & D_3 & D_2 & D_1 & D_0 & G_1 & G_0 & E_1 & E_0 \\
 1 & 0 & 9 & 8 & 7 & 6 & 5 & 4 & & & & & 0 & 1
 \end{array}$$

If a constant and program steps are stored in the following manner:

80	81	82	83	84	85	86	87	88	89	8a	8b	8c	8d
R	S	E	D ₃	D ₄	D ₅	D ₆	D ₇	D ₈	D ₉	E ₀	E ₁	G ₀	G ₁
C	I	N							D _s		E _s		
L	N	D											

and carry bit is generated in D₉, the program step in 80 (corresponding to D₀) will shift to the G₁ position and will never be executed.

Lost
Instruction

80	81	82	83	84	85	86	87	88	89	8a	8b	8c	8d
S	E	D ₂	D ₃	D ₄	D ₅	D ₆	D ₇	D ₈	D ₉	E ₀	E ₁	G ₀	R
I	N								D _s		E _s		C
N	D												L

We've therefore lost a program step if a carry is generated in D₉. All you have to do to avoid this is initially put 1000 (for instance) in the register and count by one, now carries are generated in D₆ and propagate to D₇. Carrys cause a change in the value of E₁, but neither data nor instructions are stored there. You'll never count past 9999 when a carry would be generated in D₉. You must manually enter zeroes in digit locations up to and including D₈, enter 1 in D₉ and 03 in E₀ and zero in E₁. That is one limitation, though quite easily overcome.

The second limitation is more serious. Any instruction with an octal code in the 20's, 30's, 60's or 70's cannot be used. This is because the 20 bit (F24) in the 40-20-10-4-2-1 octal system is lost except in D₉ and E₁. F24 contains the 20 bit if a program step is stored or the minus sign if a number is stored. When reading G₀ - D₈ the sign information is lost; the 9100A doesn't care what the sign is until D₉ is reached. So if 80, 81, 82 respectively contain octal codes 23, 34, 67 the codes after an add would be 03, 14, 47, the 20 bit being lost.

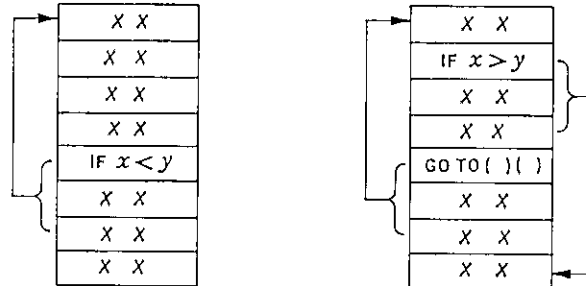
Octal Code
Limitation

Another technique which your customers may want to use is transferring a complete register of program steps using the γZ_{10} key. It can be done but again with a restriction on F24 of character location a and b in the register. F24 is the 20 bit in the octal code and F24 = 1 means there is a 20 in the code. The following table shows when F24 of character location b is lost.

	F24 = 1 in a	F24 = 0 in a
F24 = 1 in b	Lose F24 in b	OK
F24 = 0 in b	OK	OK

LOOPING

Normally, exiting a loop when the condition is false rather than true will save writing a GO TO instruction.



COUNTING

It is often necessary to use a variable to count the number of times that the program executes a series of instructions. (loop) The variable is usually increased or decreased by one each time the loop is executed. When the variable reaches a pre-determined value generally the program will branch to an address out of the loop. Here are two counting routines. The first increments the variable by one until it equals N and then branches.

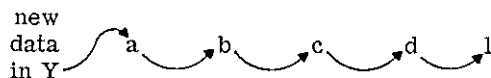
STEP	KEY	x	y	z
0-0	CLEAR			
1	STOP	n		
2	↑		n	
3	1	!		
4	ACC +			
5	f			
6	IF x < y			
7	0			
8	3			
9	END			

Uses only one-half of vector accumulator.

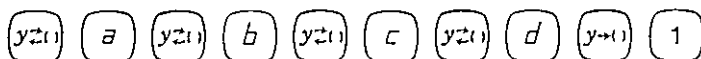
This routine will decrement from N back to zero.

STEP	KEY	x	y	z	f
0-0	CLEAR				
1	STOP	n			
2	x → ()				
3	f				n
4	1	!			
5	ACC -				
6	RCL				
7	IF x > y				
8	0				
9	4				
a	END				

This technique is useful in applications such as a moving average in which the entry of each subsequent value displaces the contents of every register in the sequence:



If the new data is in the Y register, this series of instructions will accomplish the shifting:



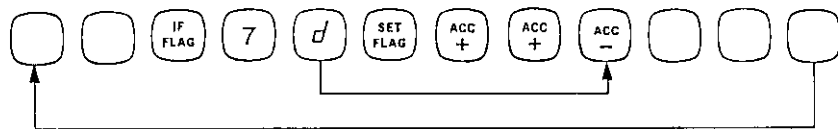
DATA REGISTER SHIFTING

Occasionally, one's program is so long he needs more than the sixteen registers for program instructions and data storage. A series of programs may be linked together by storing intermediate results and entering the next program stage from a magnetic card. Using this technique there is no limit to the number of program steps available. Remember the Calculator records instructions from memory onto a magnetic card until an END statement is encountered. Be sure the last instruction you want recorded is an END instruction.

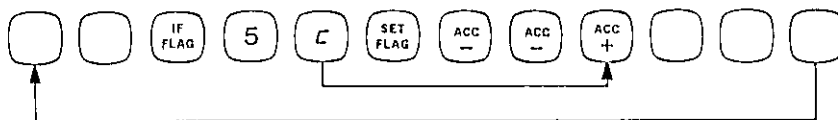
LINKING PROGRAMS

In the sum of many terms such as an arithmetic series, it is often necessary to alternate signs before adding the next term. In this series, the flag is not set on the first pass.

If the first term is (+) use:



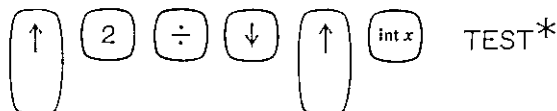
If the first term is (-) use:



TEST FOR EVEN OR ODD NUMBERS

For the first pass through the loop the flag is not set. Here is a useful technique for testing whether a number is even or odd:

Enter number in X register



* IF X = Y tests true if even
IF X < Y tests true if odd

COMPUTING SINE θ AND COSINE θ

A quick method for computing both $\sin \theta$ and $\cos \theta$ is:

Enter θ in Y register
Enter 1 in X register
Depress TO RECT

The Y register contains $\sin \theta$ and the X register contains $\cos \theta$

INSERTING PROGRAM STEPS

It is possible to insert several program steps in the middle of an existing program without re-writing the entire program. It may be done in one of two ways. As an example refer to the prime number program (follows). There are two PAUSE statements in the program: Step 2c and Step 6b. For the following examples it is convenient to work with the latter.

Example 1

In this case replace Steps 6b, c, d with GO TO (8) (4) in PROGRAM mode. Flip mode switch to RUN and GO TO (8) (4). Flip mode switch to PROGRAM and insert desired modification - PSE, PSE, PSE, PSE. Steps 6c (YE), 6d (a) may now be entered finishing the program with GO TO (7) (0), END.

Aesthetically this modification leaves something to be desired, however, there is no interference with loops within the original program. The only concern is that the added steps must not over flow into data storage register.

Example 2

With mode switch in RUN, GO TO (6) (c), enter blank card and depress RECORD. We now have Step 6c through 84 recorded on the card. Go to 6c and flip mode switch to PROGRAM entering PSE, PSE, PSE. The last PSE is in Step 70. Flip mode switch to RUN and GO (7) (1), insert card containing 6c through 84, and depress ENTER. The entire program may now be recorded on a card. One must make certain that there are no branch designations in the shifted portions of the program. If this is the case the origins must be modified accordingly.

PRIME NUMBERS

6	8	1	01
	9	↓	25
	a	e	12
	b	PAUSE	57
	c	y _z (1)	24
	d	a	13
7	0	1	01
	1	+	33
	2	y _z (1)	24
	3	a	13
	4	y _z (1)	24



CHAPTER V
C O M P E T I T I O N

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Wang stresses their building block approach to solving customer problems. The building blocks are not all compatible and increasing calculating capability usually involves replacing the keyboard and often the arithmetic unit itself. Additional storage must be installed separately for each keyboard; it is expensive and the cost runs up quite fast as you attempt to match the 9100A's capability.

WANG BUILDING BLOCKS

Time-sharing is both inefficient and expensive. Only a small part of the system is actually shared. Keyboards, data storage, and program storage are not shared. The only part shared is their equivalent of our X and Y registers!

WANG TIME SHARING

Sell the 9100A! Stress these features:

1. Trig function keys
2. Magnetic card programming
3. Dynamic range
4. Floating point
5. Ease of programming

Each model number represents only one part of a system, so several model numbers must be considered together when comparing Wang with the 9100A.

WANG MODELS

Arithmetic Units (Electronics Package)

300E	Standard calculator, basic arithmetic operations.	\$1,240
300SE	Standard calculator, basic arithmetic operations with connections for four remote time-share terminals (simultaneous electronics).	2,480
310E	Standard calculator, basic arithmetic operations with $\sqrt{x}$ and x^2 ability. Aimed at statistical applications.	1,435
310SE	Basic arithmetic with $\sqrt{x}$, x^2 ability and "simultaneous electronics" capability.	2,870
320E	Arithmetic, $\sqrt{x}$, x^2 , e^x and $\log_e x$ ability.	1,625
320SE	Arithmetic, $\sqrt{x}$, x^2 , e^x , $\log_e x$ and time-share ability.	3,250

360E	Arithmetic, $\sqrt{x}$, x^2 , e^x , $\log_e x$, with 4 storage registers.	\$1,995
362E	360E with 12 storage registers.	2,295
<u>Keyboards</u>		
300K	Standard calculator keyboard arrangement for use with 300E.	450
310K	Standard calculator keyboard with $\sqrt{x}$ and x^2 keys. For use with 310E.	460
320K	Calculator keyboard with $\sqrt{x}$, x^2 , $\log_e x$, and e^x keys. For use with 320E.	470
320KT	$\sqrt{x}$, x^2 , e^x , $\log_e x$ keys with built-in wired programs for $\sin x$, $\cos x$, $\arcsin x$, and $\arctan x$.	920
320KR	320KT for angles measured in radians.	920
360K	$\sqrt{x}$, x^2 , $\log_e x$, e^x keys (no trig keys) with 4 additional storage registers. Store_0 , Store_1 , Store_2 , Store_3 keys as well as Recall_0 , Recall_1 , Recall_2 , Recall_3 .	500
360KT	360K with four trig keys as on 320KT.	950
370, 370-2	Programming keyboard (370-2 for use with 362E). Performs standard keyboard data entry and display and provides sequencing logic for card reader programs. Card programming. No trig functions on keyboard.	1,200
380, 380-2	Essentially the 370 keyboard except that programs are stored on magnetic tape cartridges. No trig functions on keyboard.	1,500

Card Readers

CP-1	Reads 80 step, two digit octal codes. Used with 310K, 320K, or 360K keyboard.	800
------	---	-----

371	Reads 80 step program card. For use with 370. As many as six card readers may be linked together (at \$800 each) allowing up to 480 program steps.	\$ 600
<u>Data Storage</u>		
372	16-10 digit registers.	1,000
373	64-10 digit registers.	2,500
<u>Output Typewriter</u>		
379-5 & 7	IBM Selective Typewriter	1,500
376	33ASR teletype with typewriter keyboard, paper tape reader, and tape punch. (needs 377 control unit).	1,400
377	Control unit for 376.	1,400
<u>External CRT Display</u>		
379-9 & 10	CRT display and control unit. 8 cm x 10 cm screen.	2,500
<u>Trig Pack</u>		
379-12	Provides sin, cos, arc sin, arc tan, and π in two keystrokes. May be used only with Series 370 and 380 systems. Trig functions are available in all four quadrants.	1,200
<u>Card Reader Interface</u>		
379-13	Up to six 371 card readers may be interfaced to a 380/380-2.	800
<u>External CRT Display</u>		
379-9/10	5 1/2" x 8 1/2" screen. At first glance this appears impressive. Closer examination, shows that it is quite slow, has no X-Y scale, and a photograph is required to obtain hard copy.	2,500

Interface

379-8

\$2,500

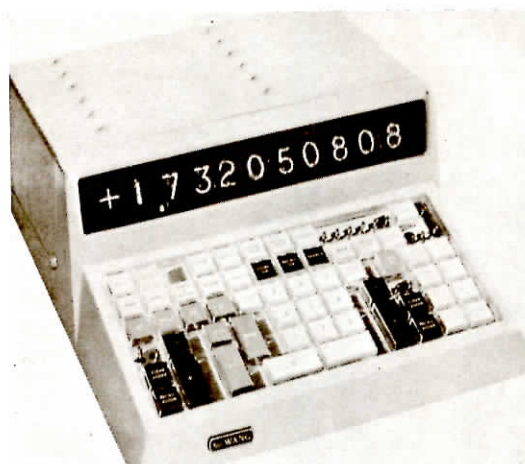
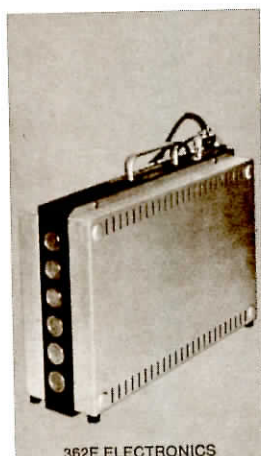
Accepts data from digital voltmeters, electronic counters, etc., for the operation of 370/380 calculating systems with measuring instruments.

Seem complicated? That's because it is!

RECAP

<u>TYPE</u>	<u>FUNCTION</u>	<u>MODEL</u>
Arithmetic	Arithmetic operations, including +, -, x, ÷, x^2 , x , $\log_e x$, e^x . SE= simultaneous operation, up to 4 keyboards. 362E has 12 storage registers.	300E or SE 310E or SE 320E or SE 360E 362E
Arithmetic Keyboard	Data entry and display. Arithmetic operations	300K 310K 320K, KT or KR 360K, KT or KR
Card Reader	For use in conjunction with keyboards, each reader holds an 80 - step program card.	371 CP-1
Programming Keyboard	Data entry and display, learn and operate programs.	370 380 380-2
Data Storage	Store input data, intermediate or final results, 10 digits/register. Model 372 has 16 registers, Model 373 has 64.	372 373
Output Typewriter	Type contents of display register in proper columnar format with alphabetical headings and messages. 376= Teletype #33ASR, 377 is its control unit; 379-5 = IBM Selectric with control unit.	376/377 379-5

The following show a typical Wang electronics pack and keyboard.



WEAK POINTS

Dynamic Range

Dynamic range from e^{-40} to e^{40} , approximately $.43 \times 10^{-17}$ to 2.35×10^{17} . The effective dynamic range is 10^{-9} to 10^9 .

Fixed Point

Because fixed point notation is used, numbers less than one lose significance seriously. (Wang "automatically" places the decimal point in the correct position--if you don't exceed 10^9 -- and calls this "floating point".)

- a) $(.00001234)^2 = .0000\ 0000\ 9$ on Wang, should be 1.522756×10^{-10}
 b) $\left(\frac{1}{(9876)^2}\right)^{-1/2} = 9853.29$ on Wang, should be 9876.

Overflow

Overflow is indicated by a flashing decimal point not easily noticed. Unpredictable answers result. The same flashing decimal point indicates that you have exceeded 10^9 and should move the decimal point over ten places to the right. You then must check to see if you are still within the 10^{17} constraint. So for all practical purposes, the upper limit is really 10^9 !!

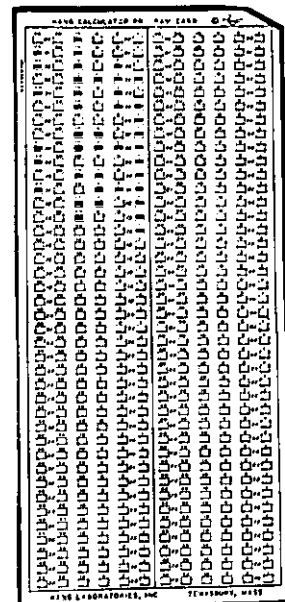
Difficult Programming

Tedious programming! An IBM porta-punch is used to punch operator codes into the program cards.

Unless a program is used fairly frequently, it not worth the time and effort of putting it on a card. And because these cards are prescored for pencil punching, there is the danger that after repeated use one of the unpunched positions will drop out, creating an incorrect code.

Example: The operator code for multiplication is 46. The card would be punched so that the sum of the weights for the punched holes totals 46.

40	20	10	4	2	1
*			*	*	



Hang Ups

The wrong sequence of keying at any one of the four keyboards which can share the calculating electronics package can cause a hang up which requires going to the location of the electronics package and pushing the prime button. Hang ups spoil the calculation at all 4 output channels.

Trig Functions

No trig keys on 370 and 380 keyboards. A card for trig functions must be read by the 370, and a tape subroutine is needed for the 380. The 379-12 Trig Pack can be added for \$1,200 to either keyboard.

Trig functions on the KT keyboards are for $0^\circ - 90^\circ$ only. Their accuracy is 10^{-8} for sin and cos, 10^{-5} for arc sin and arc tan. Trig functions are derived by a series expansion and computation time is about 3.5 seconds.

Time Sharing

"Time-sharing" is used loosely by Wang. "T" connectors automatically sequence interlinked keyboards on a basis of first-on-first-served and registers cannot be shared. Some pre-arrangement must be made by users as to which register may be used at a given time. And only one trig keyboard per system is allowed. Also, there is only one displayed register.

Installation costs for remote terminals could be \$1,000 or more. Connectors are \$25 and cable costs \$1/foot. Labor for installing cable in an area not previously designed for calculator cable is quite high.

Slow magnetic cartridge. 18 program steps per second. The Wang 380 system cannot store program instructions internally (no core storage!). The 380 system is offered with three different program lengths. The longest is 640 steps. Individual steps are addressable, but it may take as long as 34 seconds to search the tape and locate a step on the long tape. A customer would be wise to use the shortest possible tape on any given program. The speed of the tape cartridge cannot be varied, so program speed is limited by the slowest operation available. This suggests that the trig keys will not be incorporated on the keyboard directly but will remain as subroutines in a separate package.

Speed

Small register size. 10 digits plus sign and decimal point.

Separate trig keyboard is required for angles measured in radians. (KR keyboards). Again, first quadrant only.

Program correction on the 380 tape system is awkward. Tape marks can identify blocks of tape, but a particular step cannot be directly addressed. By using the tape marks and display program key, it is possible to locate a particular step. This is fine if you are certain which step is to be corrected. If not, after displaying the step, you must proceed all the way around the tape loop again to correct it. (As in 9100A, once you display the step you must re-address the step to change it.) Card programs are corrected by patching up the holes and repunching or repunching the entire card.

No rectangular coordinates to polar coordinates keys.

No hyperbolic functions keys.

Test zero. The Wang test zero operation tests for a zero only in the most significant digit position. To test 10^{-7} , the machine must loop through the program steps seven times testing each digit.

Wang's qualifiers consist of test zero, test overflow, and test positive. Only test positive is available on the keyboard and toggle switches must be used for the others. This is a little clumsy. Toggle switches are also used for indirect addressing.

Toggle Switches

STRONG POINTS

Cheaper configurations for users who may not need trig functions, qualifiers.

Building block approach. If the prospect can clearly define exactly what capability he needs, he can buy only that configuration.

Time-sharing. Four keyboards on one electronics package could be economical.

Printing capability.

Wang obtains scientific notation by storing the base in one-half register and the exponent of ten in another. This enables them to significantly increase their operating range and display results in floating point. To do this they use logarithms. The 9100A can be programmed in a similar manner to increase its dynamic range.

Subroutine capability. The Wang 370 and 380 make use of Search & Return and Return operations to insert subroutines in programs. The program counter records the step number to be executed after execution of subroutine. This counter makes it possible for the same subroutine to be used in several places in the program without having to write it over each time. This is a very useful feature.

File protection is available for the 380 magtape cartridge.

Step program is available in either the program or run mode on the 380. In addition, when stepping through a program in the program mode the octal code of each step is displayed and at the same time the instruction is executed as if it were in the run mode.

Programming

380 tape programming is accomplished by keying in the steps directly from the keyboard, as on the 9100A. No octal code conversion, such as on their card programming models, is required.

Indirect addressing of 373 auxiliary data storage. The 373 can be addressed indirectly through a pair of registers identified as the row and column counters. Store Indirect and Recall Indirect commands allow sequential storage or recall of data row by row, column by column, or in lines diagonally across the 8x8 memory matrix. This is particularly useful for matrix manipulation.

MINIMUM WANG CONFIGURATIONS

1. Roots of Functions (09100-70014)
$$f(x) = x^3 - 16x^2 + 73x - 90$$
$$360E + 370 + 372 + 371(3) = 5,995$$
2. Traverse Closure (09100-74001)
$$360E + 370 + 371(2) + 372 = 5,395$$
3. Mean, Variance, Standard Deviation
$$360E + 370 + 371 = 3,795$$

4. Linear Correlation (09100-70803)
 $360E + 370 + 371 = 3,795$
5. Linear Regression (-70803)
 $360E + 360K + 371 = 3,295$
6. Solution of n Simultaneous Equations
 (2 to 7 equations)
 $320E + 370 + 371(2) - 373 = 6,925$
7. Roots of functions (-70014)
 $360E + 370 + 371(3) + 372 = 6,595$

Dr. An Wang, President, was born in China. A harvard graduate, he started Wang Labs in 1951. Dr. Go Yao Chu, Vice President, was also born in China and received his Doctor's Degree from M.I.T. Wang Labs is located at Tewksbury, Mass. and has about 400 employees. 1966 sales were \$2.6 million; in 1967 their sales were \$6.7; 1968 sales should be about \$15 million. Warner and Swasey Company own 25% interest in Wang. The Wang family owns 63% of the stock which was first publicly sold in 1967. The price of the stock has ranged from 12 1/2 to 81.

WANG
COMPANY

Mathatron is another company that sells time-sharing as an important feature. However, there is no program storage capability at the remote terminals, so that a program must be keyed in every time it is used. Mathatron is not programmable in the usual sense. It has no capability for conditional branching or compares. Formula storage is their term.

MATHATRON

Sell the 9100A! Stress these features:

1. Programmable.
2. Dynamic range.
3. Low cost.

PKB keyboard and Central Station

\$6,750

PKB

Keyboard	950
Central Station	3,450
Expanded Memory	3,300
Page Printer Control	950
Page Printer (8 1/2")	1,850
Perforated Tape Reader	475
Perforated Tape Punch	150



PKB \$6,750



4280A \$9,990

GOOD FEATURES

There are good features that you should be aware of:

1. Central Computing Station which can be connected to remote keyboards using a telephone connection or a standard two wire connection.
2. Serial strip printer.

3. Floating point and enter exponent key.
4. Power log key. This key raises a number to a whole or fractional, positive or negative power, with the use of logs or exponentials.
5. Parenthesis. These are keyed directly as they appear in the equation.
Example:

$$(513 \div 64) \div (8 \times 14.7) (1.416 \div 3) = .144\ 541\ 421$$
 Keystrokes:

$$(513 \div 64) \div (8 \times 14.7) (1.416 \div 3) = 27 \text{ key-strokes}$$
 This problem takes 33 keystrokes on the 9100A.
6. Expanded memory allows conditional branching, additional storage. Expands formula storage to 480 algebraic steps. The Central Station has only 8 registers.
7. Four separate keyboards available.
 - a. Surveyor. Keys: arc tan, radians to degrees, sin, cos, bal. The trig functions work in all four quadrants.
 - b. Scientist. Keys: log, $\tan^{-1}$, sin, $s(x^2)$, Σ , r (correlation coefficient), t.
 - c. Statistician. Keys: Σ , $s(x^2)$, r, t. PDA (paired data analysis).
 - d. Mathematician. Keys: log, 3x3 (determinants), 25 (two simultaneous equations), sin, $\tan^{-1}$.

These are some limitations you might keep in mind:

1. No conditional branching or compares.
2. Range 10^{-42} to 10^{+58} .
3. Slow.
4. No way to store program from remote keyboard.
5. Remote keyboards are not simultaneous.

Mathatron was acquired by Barry Wright Corporation in 1965. Mathatron's sales are about \$2 million annually, employing about 100-150 people and lost money in 1967.

LIMITATIONS

Olivetti has a large sales force and a printer. The machine was designed for business applications. It has no engineering keys, no scientific notation, limited range, is difficult to program, is noisy and slow.

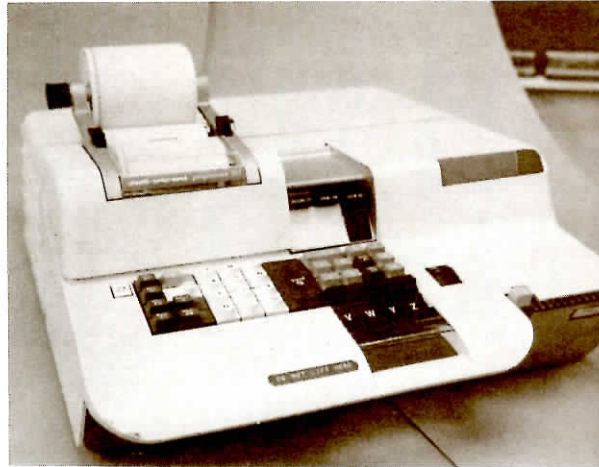
Sell the 9100A! Stress these features:

1. Scientific keyboard.
2. Dynamic range.
3. Easy Programming.
4. Floating point notation.

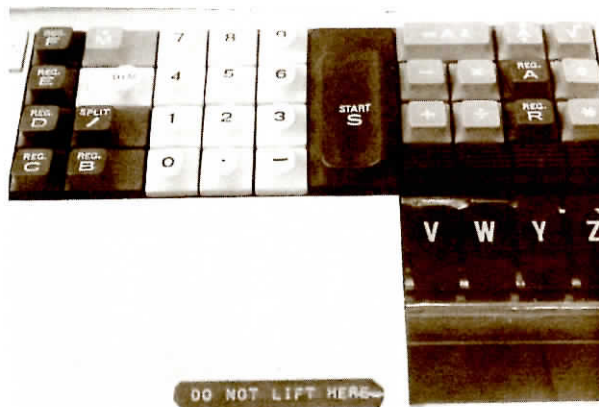
The Olivetti Programma 101 has a $\sqrt{\quad}$ key, 10 storage registers, and a program capacity of 120 instructions. The Split / key divides a

OLIVETTI

register into two equal parts. The V, W, Y, and Z keys direct the machine to a particular program or subroutine. The free registers have a capacity of 22 digits plus decimal point and sign. Programs are stored on 9 1/2" x 2 3/4" magnetic cards. It has printed output on a paper tape roll. The Programma sells for \$3,850.



Programma 101



Keyboard

DISADVANTAGES

The sequence of steps required for the simplest operations are very cumbersome. The Olivetti representative prefers to work from program cards and doesn't like to write programs in front of the customer.

Not only is the P101 difficult to program there is no display of intermediate results so that you don't know until you are through the problem (or get a lock up) if you are proceeding properly. And the absence of the CRT makes programs extremely difficult to debug.

The 22 digit capacity, fixed point notation, is a severe limitation. Olivetti calls their "automatic" decimal point "floating point", but it is not scientific notation.

EXAMPLE: Set decimal wheel at 15. Divide 1 by 10^{-4}

- a. Enter 1. This will take 15 storage digits.
- b. Enter .0001. This will take 4 storage digits.
- c. Divide. Result takes 4 storage digits.

The total of 23 storage digits exceeds the capacity of the machine and it locks up. By the same rule one cannot multiply two 12 digit numbers together and obtain an answer which would be accurate to 24 places.

There are no trig, log, polar, rectangular, hyperbolic keys. Any operation that is not arithmetic, except $\sqrt{\quad}$, must be programmed as a separate subroutine.

The Olivetti representative will turn off his machine during a demo because you cannot talk above the noise. And because of no power protect capability, he casually re-enters his program when he turns the machine back on.

It takes 2 or 3 seconds from the time a card is put into the card feeder until it is loaded into the machine. To calculate the sine of 1.57 R takes 37 steps, 27 seconds.

Often in a program an entire register is inefficient for storing a small constant, such as 31.68. In these cases, the digits and decimal point would be entered as separate program steps (3, 1, ., 6, 8) requiring 5 steps instead of 14 steps of a full register. This is referred to as "generating constants". Olivetti's method for this is unbelievable! Constants themselves cannot be entered. Program instructions representing constants are used. Not only that, but they must be entered in reverse order! Beautiful! Olivetti's generation of 31.68 would be:

A/ $\uparrow$ To start special series of instructions
B $\diamond$ 8
RX 6
R/ $\downarrow$ 1.
D $\updownarrow$ 3 and termination of special instructions.

The P101's qualifier capability is quite limited and very awkward. Comparison in the 9100A manner, $X > Y$, is not possible. The only straight forward comparisons that can be made are $X > 0$ and $X < 0$. The machine cannot directly differentiate between zero and negative numbers. This distinction is a two step procedure.

CONSTANT GENERATION

POINTS
STRESSED
BY
OLIVETTI

Their printer is certainly their strongest and most persuasive point.

Olivetti makes a big deal out of their ability to split registers with a keyboard key. However, if you split a register the number which is stored is seriously limited, and you must be extremely careful that the figures you put in do not exceed the eleven digit capacity. Numbers of less than one lose significance seriously. For example, $1.23456789 \cdot 10^{-15}$ would be stored as .00000012345. If you exceed the capacity of the machine you will cause a lock-up. If you exceed the split register capacity enormous errors occur and the machine must then be cleared and the entire problem started again.

Olivetti publishes a 281 page book of subroutines, or general utility programs, to aid programming which, on the surface, appears quite impressive. There are 63 of these subroutines. (The numbering goes to 167, but the numbering is not consecutive). Of these, 26 are merely keystrokes on the 9100A! Six other subroutines are for solving the exponential b^n for various n 's. So over half of the Olivetti subroutines are for performing operations that can be keyed on the 9100A!

SUBROUTINE
LIMITATIONS

There are several serious limitations to using Olivetti's subroutines in a program.

Use of subroutines results in a significant reduction in the number of program steps available for the rest of the problem solution.

An error is introduced by conflicting decimal wheel settings. For decimal wheel setting at 10, subroutine 41 for $\sin X$ has an error of 7×10^{-5} ; subroutine 22 for e^x has an error of 10^{-10} . So you're not sure what the error will be for various wheel settings, especially if these subroutines are used together.

Argument range severely limits the practical utility of any program containing a subroutine, particularly with the log and exponential functions. It is difficult to examine a relatively complex problem with a varying exponent or log routine and determine the range of these functions. If the argument range is exceeded the 101 will continue to try to use the subroutine and will produce gross errors in the derived function without any indication to the operator unless an overflow condition or illegal operation is encountered.

Most of their subroutines are slow. Subroutine 41 for $\sin X$, decimal wheel at 10, takes 6 seconds. Subroutine 42 for $\sin X$, decimal wheel at 10, is more accurate than 41 (error of 10^{-10}) but takes 28 seconds.

OLIVETTI
COMPANY

Olivetti SpA is an Italian company that is the leading European producer of office machinery and equipment. Olivetti owns 91% of Olivetti-Underwood which represents them in the U.S. World-wide sales amounted to \$546 million in 1967. Olivetti-Underwood earned \$3.4 million on \$157 sales that year. Olivetti employs 58,000 people, of which 26,000 are in Italy.

1. You are not able to raise a number x to a power y (x^y) without using logs. This feature is only available as x^2 on Wang and not at all on Olivetti.
2. Floating point is limited to a number with a maximum of one digit to the left of the decimal. 10×10^6 must be entered as 1, ENTER EXP, 7 (1×10^7).
3. A given subroutine can be used twice in a program by making use of the flag. Wang's Search & Return allows the subroutine to be used over and over. The customer must decide how many times he really needs subroutines in a program and whether he is willing to sacrifice all of the benefits derived from the 9100A use for this one Wang feature.
4. The error light does not indicate overflow or underflow. 1×10^{104} would be displayed as 1×10^4 .
5. The 9100A does not switch from fixed point to floating point when the quantity in the X register underflows the display, based on the decimal wheel setting. Though this feature is not available on any other calculator, the customer might expect it to be on the 9100A.

SHORTCOMINGS
OF
9100A

MODELS	KEYS											sin	cos
	$\sqrt{x}$	x^2	log	ln	e^x	VECTOR	π	$ y $	int x	HYPER			
WANG 300K 300E													
310K 310E	X	X											
320K 320E	X	X		X	X								
320KT 320E	X	X		X	X						I	I	
360K 360E	X	X		X	X								
360KT 360E	X	X		X	X						I	I	
380-2 362E	X	X		X	X								
OLIVETTI PROGRAMMA IOI	X												
WANG 370 320E 371 372	X	X		X	X								
380-2 362E 372	X	X		X	X								
hp 9100A	X	Q	X	X	X	X	X	X	X	X	ALL	ALL	
WANG 380-2 362E 379-12	X	X		X	X		d				d	d	
360KT 360E 371(2) 372	X	X		X	X						I	I	
370 320E 371(2) 373	X	X		X	X								
360KT 360K(3) 371(4) 372 360E	X	X		X	X						c	c	
370 320E 371 373 376 377	X	X		X	X								
380(4) 360SE 376 377 379-12	X	X		X	X		d				d	d	
380(4) 360SE 373 376 377 379-12	X	X		X	X		d				d	d	

a. x^2 is 2 keystrokes on 9100A.

b. 9100A program steps are corrected by addressing the particular step to be changed in Run mode, switching to Program mode, and entering new (correct) step.

c. "Time-Sharing" limits trig functions to one keyboard, 1st quadrant only.

d. Wang 379-12 Trig Pack makes and trig functions available in two keystrokes. Trig functions in 4 quadrants.

e. Wang splits registers for base and exponent to work in scientific notation.

f. 380 program steps are awkward to find but can be corrected from keyboard.

QUADRANTS FOR
TRIG KEYS

$\tan$	$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$	STORAGE REGISTERS	SCIENT NOTATION	DYNAMIC RANGE	LOOP BRANCH	PROGRAM TYPE	PROGRAM STEPS	MODE FOR STEP PROGRAM	CORRECT PROGRAM	TIME SHARE TERMINALS	INDIRECT ADDRESSING	DISPLAY	PRINT OUT	COST
					e	$10^{-10} \rightarrow 10^{10}$								1 REG.		1,690
					e	$10^{-10} \rightarrow 10^{10}$								1 REG.		1,895
					e	$10^{-10} \rightarrow 10^{10}$								1 REG.		2,095
	I		I		e	$10^{-10} \rightarrow 10^{10}$								1 REG.		2,545
				4	e	$10^{-10} \rightarrow 10^{10}$								1 REG.		2,495
	I		I	4	e	$10^{-10} \rightarrow 10^{10}$								1 REG.		2,945
				12	e	$10^{-10} \rightarrow 10^{10}$	x	MAG TAPE	640	PROG RUN	f			1 REG.		3,795
				10		22 DIGITS	x	MAG CARD	120/SIDE PER CARD						x	3,850
				16	e	$10^{-10} \rightarrow 10^{10}$	x	PUNCH CARD	80		g			1 REG.		4,425
				28	e	$10^{-10} \rightarrow 10^{10}$	x	MAG TAPE	640	PROG RUN	f			1 REG.		4,795
ALL	ALL	ALL	ALL	19	x	$10^{-98} \rightarrow 10^{99}$	x	MAG CARD	196/SIDE PER CARD	PROG RUN	b			3 REG.		4,900
	d		d	12	e	$10^{-10} \rightarrow 10^{10}$	x	MAG CARD	640	PROG RUN	f			1 REG.		4,995
	I		I	20	e	$10^{-10} \rightarrow 10^{10}$		PUNCH CARD	160					1 REG.		5,145
				64	e	$10^{-10} \rightarrow 10^{10}$	x	PUNCH CARD	160		g			1 REG.		6,725
	c		c	20	e	$10^{-10} \rightarrow 10^{10}$		PUNCH CARD	80		g	4		1 REG.		7,845
				64	e	$10^{-10} \rightarrow 10^{10}$	x	PUNCH CARD	80		g		x	1 REG.	x	8,725
	d		d	4	e	$10^{-10} \rightarrow 10^{10}$	x	MAG TAPE	640	PROG RUN	f	4	x	1 REG.	x	13,990
	d		d	68	e	$10^{-10} \rightarrow 10^{10}$	x	MAG TAPE	640	PROG RUN	f	4	x	1 REG.	x	16,490

g. Patch existing card or make a new card.

CHAPTER VI
ENGINEERING MATHEMATICS

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The roots of a function $f(x)$ are the values of x which make $f(x) = 0$. The roots (solutions) of the quadratic equation $ax^2 + bx + c = 0$ are:

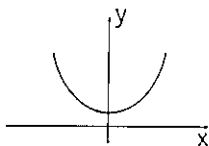
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $\Delta = b^2 - 4ac$ is called the discriminant.

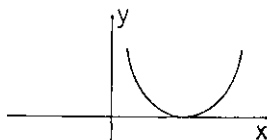
- a. If $\Delta > 0$, the roots are real and distinct;
- b. If $\Delta = 0$, the roots are real and equal;
- c. If $\Delta < 0$, the roots are complex.

Graph and find the roots of the following quadratic equations on the 9100A. Use program 09100-70009.

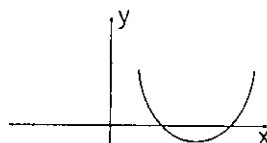
$$y = x^2 + 1 \quad -3 < x < 3$$



$$y = x^2 - 4x + 4 \quad 0 < x < 4$$



$$y = x^2 - 5x + 6 \quad 0 < x < 4$$



Evaluate the polynomials between $x = -3$ and $x = +4$ with the polynomial evaluation program number 09100-70002 and check the roots.

The symbol $n!$, called "n factorial", stands for the product of the first n (positive) integers.

$$n! = 1 \cdot 2 \cdot 3 \cdots n;$$

$$0! = 1, \text{ by definition}$$

Evaluate $50!$ using program 09100-70001. Try $71!$ What is the maximum n before overflow on the 9100A? What is the maximum n before overflow on the Olivetti? Does the 9100A warning light come on?

QUADRATIC EQUATION

FACTORIAL NOTATION

GAMMA FUNCTION

Notice the argument of the factorial must be an integer. There is also a generalized factorial function which is known as the gamma function:

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx \quad n > 0$$

The gamma function may have any value for its argument. For integer values it is related to the factorial

$$\Gamma(n+1) = n! \quad n = 1, 2, 3, \dots$$

BINOMIAL THEOREM

The series expansion of $(a+b)^n$, where n is a positive integer, is

$$(a+b)^n = a^n + na^{n-1}b + \dots + \frac{n(n-1)}{2!} b^2 a^{n-2} + \dots + b^n$$

LOGARITHMS

If $a^b = x$, then, by definition of logarithms, $\log_a x = b$

$$\log_b a = \frac{1}{\log_a b}$$

$$\log MN = \log M + \log N$$

$$\log M^A = A \log M$$

$$\log \frac{M}{N} = \log M - \log N$$

$$\log M^{\frac{1}{A}} = \frac{1}{A} \log M$$

Suppose a customer notices there is no antilog_{10} key on the 9100A. How would you find this using available functions? Given Y solve for X where:

$$Y = \log_{10} X$$

What is the antilog_{10} 15.565?

Answer: 3.6728×10^{15}

Relation between degree measure and radian measure:

$$360^{\circ} = 2\pi \text{ radians} = 1 \text{ revolution}$$

Convert 32.2° to radians. Can you think of a way to do this with only 3 key depressions? Try converting 135° to radians.

Let a, b, c be sides; A, B, C , opposite angles:

$$s = \frac{a + b + c}{2} \quad \text{semi-perimeter;}$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \quad \text{radius of the in-scribed circle;}$$

If R is the radius of the circumscribed circle,

$$\text{a. Law of sines: } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\text{b. Law of cosines: } a^2 = b^2 + c^2 - 2bc \cos A$$

$$\text{c. Law of tangents: } \frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$$

$$\text{d. Tangent of half angle: } \tan \frac{1}{2} A = \frac{r}{s-a}$$

$$\text{e. Area: } K = \frac{1}{2} ab \sin C = \sqrt{s(s-a)(s-b)(s-c)} = rs = \frac{abc}{4R}$$

Fundamental identities.

$$\text{a. } \sin^2 x + \cos^2 x = 1$$

$$\text{b. } 1 + \tan^2 x = \sec^2 x$$

$$\text{c. } 1 + \cot^2 x = \csc^2 x$$

Any trigonometric function of the angle $(k\pi/2 \pm \alpha)$ is equal to $(\pm)$ the same function of α , if k is even, and is equal to $(\pm)$ the cofunction of α if k is odd.

The "+" sign is used if the original function of the original angle $(k\pi/2 \pm \alpha)$ is positive; the "-" sign is used if the original function is negative. The sign of the original function of $(k\pi/2 \pm \alpha)$ is determined by the usual quadrant conventions. These equations are useful in reducing machine error. To summarize:

$$\text{Any function of } (k\pi/2 \pm \alpha) = \pm \begin{cases} \text{Same function of } \alpha, & \text{if } k \text{ is even;} \\ \text{cofunction of } \alpha, & \text{if } k \text{ is odd.} \end{cases} \text{ Use sign of original function of } (k\pi/2 \pm \alpha)$$

RADIANS

PLANE
TRIANGLES

TRIGONOMETRY

ANALYTIC GEOMETRY

THE DERIVATIVE

Functions of the sum and difference of two angles.

- $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
- $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
- $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

Sum and product formulae.

- $\sin x + \sin y = 2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)$,
- $\sin x - \sin y = 2 \cos \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y)$,
- $\cos x + \cos y = 2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)$,
- $\cos x - \cos y = -2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y)$,
- $\sin x \sin y = \frac{1}{2} \cos(x-y) - \frac{1}{2} \cos(x+y)$,
- $\sin x \cos y = \frac{1}{2} \sin(x-y) + \frac{1}{2} \sin(x+y)$,
- $\cos x \cos y = \frac{1}{2} \cos(x-y) + \frac{1}{2} \cos(x+y)$.

Note the savings in programming steps when using the left side of these equations.

- Distance between two points:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

- Slope of line through two points

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

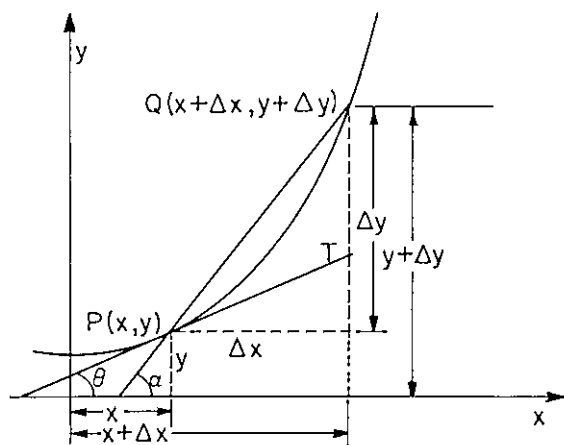
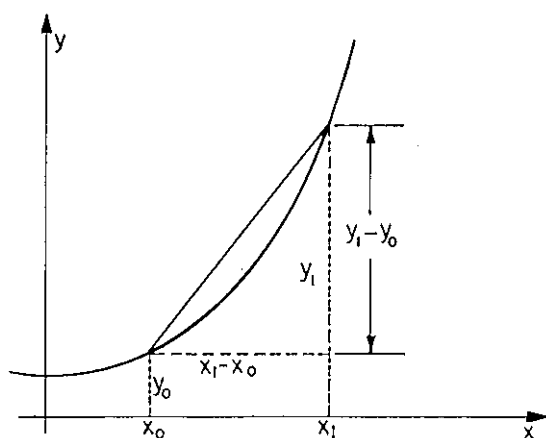
Find the equation of the straight line passing through (1,1) and (3,7).
Find the distance between these two points.

The distance between two points in three dimensional space is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The essence of differential calculus is the concept of the derivative of a function, the fundamental ideas and the definition are as follows:

Consider a function $y = f(x)$. The value of this function, at $x = x_0$, is $y_0 = f(x_0)$, and at $x = x_1$ is $y_1 = f(x_1)$. Top figure next page.



Suppose x_0 is a fixed point and x_1 is a variable point. Let x_1 approach x_0 . The derivative of $f(x)$ with respect to x at the point x_0 is defined as

$$\lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

More frequently the following notations are used. Let $x_1 - x = \Delta x$, $f(x_1) - f(x) = \Delta y$; Δx and Δy are called increments of the independent and dependent variables respectively. Then the derivative of y with respect to x is written

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Various symbols are in constant use for this limit:

$$\frac{dy}{dx} = \frac{df}{dx} = y' = f' = D_x y = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

INTEGRATION

Examine lower equation carefully. Note that, geometrically, $\frac{\Delta y}{\Delta x} = \tan \alpha$ where α is the angle made by the secant line PQ and the X-axis. As $\Delta x \rightarrow 0$, Q $\rightarrow$ P along the curve and hence the secant PQ approaches the tangent, the point of tangency being P. Therefore, the derivative of a function represents the trigonometric tangent of the angle between the geometric tangent to the curve and the X-axis. That is:

$$\frac{dy}{dx} = \tan \theta = \frac{\text{slope of the tangent}}{\text{line PT.}}$$

Write a program to take the derivative of some general $f(x)$ using the limiting definitions. Try e^x and check against an analytic derivative at $x = 5$ and $x = 25$.

Suppose $\frac{dy}{dx} = f(x)$ and the value of y is desired. This means y is some function whose derivative is $f(x)$. The answer is:

$$y = \int f(x) \, dx$$

Since two functions differing only by a constant have the same derivative, an arbitrary constant of integration must be added to an indefinite integral.

When a function, $f(x)$, is differentiated, it is necessary to specify the variable with respect to which the differentiation is to take place. The derivative of $f(x)$ with respect to x is $\frac{df}{dx}$, the derivative with respect to t is $\frac{df}{dt}$ or $\frac{df}{dx} \cdot \frac{dx}{dt}$.

In the reverse process of integration it is also necessary to integrate with respect to a certain variable.

$$\frac{dy}{dx} = f(x)$$

$$dy = f(x) \, dx$$

Integrating both sides

$$y = \int dy = \int f(x) \, dx$$

where the differential of x , dx , indicates integration of $f(x)$ is to take place with respect to x .

DEFINITE INTEGRALS

Suppose the function y whose derivative is $f(x)$ and whose value is y_0 when $x = x_0$ is desired. Let the indefinite integral of $f(x)$ by $F(x) + c$; then

$$y = \int f(x) \, dx = F(x) + c$$

To impose the condition $y = y_0$ when $x = x_0$ means

$$y_0 = F(x_0) + c$$

and hence c is determined:

$$c = y_0 - F(x_0).$$

Thus the solution to the problem is

$$y = F(x) + y_0 - F(x_0),$$

which reduces to y_0 when $x = x_0$. The function y is definite. There is no longer an arbitrary constant. y satisfies the initial or boundary condition $y = y_0$ when $x = x_0$. Of course, y still depends upon x .

Rearranging the previous equation

$$y - y_0 = F(x) - F(x_0)$$

Let $y = y_1$ when $x = x_1$. Then

$$\begin{aligned} y_1 - y_0 &= F(x_1) - F(x_0) \\ &= \int_{x=x_0}^{x=x_1} f(x) \, dx - \int_{x=x_0}^{x=x_0} f(x) \, dx \end{aligned}$$

This difference is the definite integral of $f(x)$; it is a constant and does not depend upon x .

The definite integral of $f(x)$ is the difference between two values of the integral of $f(x)$ for two corresponding values of the variable x . This difference is usually represented by

$$\int_{x_0}^{x_1} f(x) \, dx. \quad \text{Thus:}$$

$$\begin{aligned} \int_{x_0}^{x_1} f(x) \, dx &= F(x_1) + c - [F(x_0) + c] \\ &= F(x_1) - F(x_0). \end{aligned}$$

This is read "the integral from x_0 to x_1 of $f(x)$ "; x_1 is the upper limit and x_0 the lower limit. x_0 and x_1 are the limits of integration and the function $f(x)$ is integrated between the limits x_0 and x_1 .

$$\begin{aligned} \int_0^1 (x^2 + e^x) \, dx &= \left[\frac{x^3}{3} + e^x \right]_0^1 \\ &= (1/3 + e) - (1) \\ &= e - 2/3 \end{aligned}$$

APPROXIMATE INTEGRATION

It is impossible in some cases to express the indefinite integral, $\int f(x)dx$, in terms of elementary functions. But often, as in engineering problems, only an approximation to the definite integral

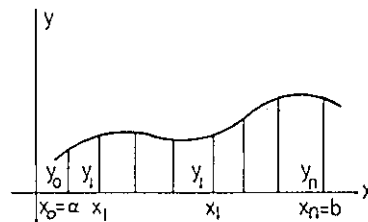
$$\int_a^b f(x) dx$$

is needed. Further, the function $y=f(x)$ may be described only by a table of values obtained through experimentation. In either case it is possible to obtain an approximate value for $\int_a^b f(x) dx$.

This definite integral may be interpreted as the area under $y = f(x)$ from $x = a$ to $x = b$. Divide the interval $b - a$ into n equal parts each of width

$$\Delta x = \frac{b - a}{n}$$

Erect ordinates y_i at points x and join the extremities by straight line segments. Then the sum of the areas of the trapezoids



thus formed is an approximation to the area under the curve. This sum is:

$$1/2 (y_0 + y_1) \Delta x + 1/2 (y_1 + y_2) \Delta x + \dots + 1/2 (y_{n-1} + y_n) \Delta x =$$

$$\frac{\Delta x}{2} (y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n).$$

TRAPEZOIDAL RULE

Hence;

$$\int_{x_0}^{x_n} f(x) dx \cong \frac{\Delta x}{2} (y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n).$$

When $f(x)$ is given as a table, this formula may be applied if the spaces between abscissa points are equal; if the spaces are unequal, the areas of the trapezoids can still be added together by using

$$1/2 (y_0 + y_1) \Delta x_1 + 1/2 (y_1 + y_2) \Delta x_2 + \dots + 1/2 (y_{n-1} + y_n) \Delta x_n$$

where

$$\Delta x_i = x_i - x_{i-1}$$

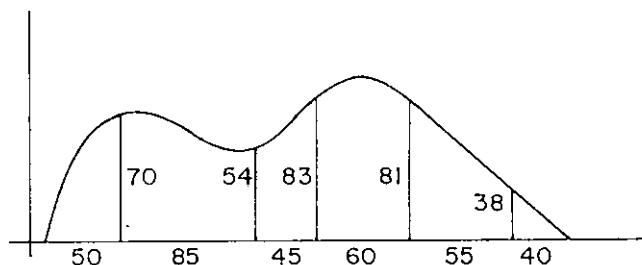
EXAMPLE: Evaluate $\int_0^2 x^2 dx$ by the Trapezoidal Rule, using $n = 4$.

Note, this can be integrated analytically and so there is no need to use a method of approximation. But as an exercise take $n = 4$ and $\Delta x = 0.5$

$$\int_0^2 x^2 dx \sim \frac{0.5}{2} \left(0 + \frac{1}{2} + 2 + \frac{9}{2} + 4 \right) = 2.75$$

The exact value is $8/3$ sq. units. The error is .08 using this numerical approximation.

In order to get an approximation to the area between straight railroad track and a winding stream, measurements were made as in the figure below. Find approximately the area between the railroad and the stream.



Solution: $A \cong 19.055$ sq. ft.

Write a program using the trapezoidal rule. Try a function described by tabulated data and a function such as $\int_0^x \sin x$. Try different increments and investigate the error.

A definite integral may also be evaluated approximately by Simpson's Rule

SIMPSON'S
RULE

$$\int_a^b f(x) dx \sim \frac{\Delta x}{3} \left[y_0 + 4(y_1 + y_3 + y_5 + \dots) + 2(y_2 + y_4 + y_6 + \dots) + y_n \right]$$

In applying Simpson's Rule it is necessary that the number of intervals n be even. The way to remember this rule is to note that when the first ordinate is called y_0 and the last ordinate y_n , n even, then the operations are =

Add the first and last ordinates

Add 4 times the sum of the ordinates with odd subscripts

Add 2 times the sum of the ordinates with even subscripts (excluding, of course, y_n).

Multiply the total sum by $\frac{\Delta x}{3}$.

This rule is developed by using parabolic arcs (instead of straight lines as in the Trapezoidal Rule) to approximate the curve. The parabolas have axes parallel to the Y-axis with each one of them passing through a consecutive set of three points on the curve of $y = f(x)$. Since each set of three points accounts for two intervals, the total number of intervals will be a multiple of 2 or an even number. In general, Simpson's Rule gives a better approximation than the Trapezoidal Rule, where the number of intervals n is about the same.

Apply Simpson's Rule to $\int_0^2 x^2 dx$

$$A = 8/3 \text{ sq. units}$$

Note that Simpson's Rule gives the exact answer in this case. Simpson's Rule gives the exact answer in case $y = ax^3 + bx^2 + cx + d$ with $n = 2$.

Evaluate $\int_0^{1.2} e^{-x^2} dx$ by Simpson's Rule, using $n = 6$.

Solution: $\Delta x = .2$, $y_0 = 1$, $y_1 = .96079$

$$y_2 = .85214, y_3 = .69768$$

$$y_4 = .52729, y_5 = .36788$$

$$y_6 = .23693$$

$$\int_0^{1.2} e^{-x^2} dx = .80675$$

Program 09100-70003 uses Simpson's rule to obtain the area under a curve.

INFINITE SERIES

When a known function $f(x)$ is written in the form of an infinite series, the function is said to be expanded in an infinite series and the series represents the function in the interval of convergence.

Since $1 + x + x^2 + \dots + x^n + \dots$ is a geometric series, it will converge to

$$\frac{1}{1-x} \text{ for } |x| < 1. \text{ Hence,}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots, \quad -1 < x < 1$$

and the function $\frac{1}{1-x}$ is expanded in an infinite series. The series representation of the function is valid only when $-1 < x < 1$, but the function exists and is continuous everywhere except at $x = 1$.

A function $f(x)$ may be expanded in a power series as follows: Write

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots,$$

where the coefficients a_i are to be determined. Then differentiating we obtain:

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots,$$

$$f''(x) = 2a_2 + 2(3)a_3x + \dots + n(n-1)a_nx^{n-2} + \dots,$$

$$f^{[n]}(x) = n!a_n + (n+1)!a_{n+1}x + \dots.$$

From these the coefficients a_i can be determined by setting $x = 0$ in each equation

$$a_0 = f(0)$$

$$a_1 = f'(0), \quad a_2 = \frac{f''(0)}{2!}, \quad \dots,$$

$$a_n = \frac{1}{n!} f^{[n]}(0).$$

This gives the expansion of a known function $f(x)$ in Maclaurin's Series which is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{[n]}(0)}{n!}x^n + \dots.$$

The expansion is said to be about the origin or in the neighborhood of the origin. To expand e^x in a Maclaurin Series we note that for all n , $f^{[n]}(x) = e^x$. Therefore the expansion is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots.$$

Expand $\sin x$ in a Maclaurin Series,

$$f(x) = \sin x, \quad f'(x) = \cos x$$

$$f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad \dots.$$

The $\sin 0 = 0$ and $\cos 0 = 1$; therefore, the expansion is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots.$$

MACLAURIN'S SERIES

e^x SERIES

$\sin x$ SERIES

This is the technique used for evaluating functions on competitive calculators. The 9100A uses much faster vector algorithms.

Write the program to evaluate $\frac{1}{1-x}$ by the Maclaurin Series expansion. Display the partial sum after addition of each term, and stop summing when $x^n < 10^{-9}$. Evaluate the series at $x = .5$. What happens when $x = 1$ and $x = 1.5$?

Write a Maclaurin's Series expansion for $\sin x$ on the 9100A. What is the range of convergence?

TAYLOR'S SERIES

A function $f(x)$ may be expanded about any point 'a' instead of about the origin. That is, $f(x)$ can be represented by a series of the form

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n + \dots$$

The coefficients a_i are computed by repeated differentiations of this relation and subsequent evaluations at the point $x = a$. Thus, we have

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \dots + na_n(x-a)^{n-1} + \dots,$$

$$f''(x) = 2a_2 + 2 \cdot 3a_3(x-a) + \dots + n(n-1)a_n(x-a)^{n-2} + \dots,$$

$$f^{[n]}(x) = n!a_n + (n+1)!(x-a) + \dots$$

From these we compute

$$a_i = \frac{f^{[i]}(a)}{i!}, \quad i = 0, 1, 2, \dots,$$

and the series expansion of $f(x)$ in the neighborhood of the point a becomes

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{[n]}(a)}{n!}(x-a)^n + \dots$$

This is known as Taylor's Series. When $a = 0$ it reduces, as a special case, to Maclaurin's series. Let $x = a + h$ and another form of Taylor's Series is obtained,

$$f(a+h) = f(a) + \frac{f'(a)}{1!}h + \frac{f''(a)}{2!}h^2 + \dots + \frac{f^{[n]}(a)}{n!}h^n + \dots$$

GENERAL FORM

SERIES REMAINDER

Let $r_n(x)$ denote the remainder of the series after n terms, or, in symbols

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{[n-1]}(a)}{(n-1)!}(x-a)^{n-1} + r_n(x) = S_n(x) + r_n(x).$$

It can be shown that $r_n(x) = \frac{f^{[n]}(x_1)}{n!} (x-a)^n$ where $a < x_1 < x$.

The Taylor's Series expansion of a function $f(x)$ will be a valid representation of the function for those values of x and only for those values of x for which $\lim_{n \rightarrow \infty} r_n(x) = 0$.

If Taylor's Series in a given instance turns out to be an alternating series, the remainder $r_n(x)$ will not exceed numerically the first term in the remainder. In general

$$r_n(x) = \frac{f^{[n]}(x_1)}{n!} (x-a)^n$$

where x_1 is some point that lies between a and x , but it is impossible to calculate $r_n(x)$ with no detailed information about x_1 . However, if x'_1 is arbitrarily chosen so $f^{[n]}(x'_1)$ is as large as possible, then

$$|r_n(x)| \leq \left| \frac{f^{[n]}(x'_1)}{n!} (x-a)^n \right|$$

The error made, therefore, in breaking off at the n^{th} term does not exceed

$$\left| \frac{f^{[n]}(x'_1)}{n!} (x-a)^n \right|$$

Let's expand $\sin x$ in a Taylor's Series about the point $\pi/6$.

$f(x) = \sin x$, $f'(x) = \cos x$, etc. These evaluated at $\pi/6$, become

$$f\left(\frac{\pi}{6}\right) = \frac{1}{2}, f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \text{ etc.}$$

Therefore,

$$\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{6}\right) - \frac{1}{2} \frac{\left(x - \frac{\pi}{6}\right)^2}{2!} - \frac{\sqrt{3}}{2} \frac{\left(x - \frac{\pi}{6}\right)^3}{3!} + \dots$$

$$-\infty < x < \infty.$$

ALTERNATING
SIGN

ERROR

Sin x Series

e^x Series

To expand e^x in powers of $(x-a)$ let

$$\begin{aligned} f(x) &= e^x \\ f^{[i]}(x) &= e^x \\ \text{and } f^{[i]}(a) &= e^a \end{aligned}$$

the expansion becomes

$$e^x = e^a \left[1 + (x-a) + \frac{(x-a)^2}{2!} + \dots + \frac{(x-a)^n}{n!} + \dots \right] \quad -\infty < x < \infty.$$

This technique may be used for faster convergence of a transcendental function not on the 9100A keyboard.

SERIES INCREMENT

Recall that, approximately, $\Delta y = f'(x) \Delta x$. Rewriting Taylor's Series provides a clearer picture. Set $a = x$, $h = \Delta x$. This gives

$$f(x + \Delta x) = f(x) + \frac{f'(x)}{1!} \Delta x + \frac{f''(x)}{2!} (\Delta x)^2 + \dots + \frac{f^{[n]}(x)}{n!} (\Delta x)^n + \dots$$

Now transpose $f(x)$ and write $\Delta y = f(x + \Delta x) - f(x)$

$$\Delta y = f'(x) \Delta x + \frac{f''(x)}{2!} (\Delta x)^2 + \dots + \frac{f^{[n]}(x)}{n!} (\Delta x)^n + \dots$$

From this, as a first approximation for small increments,

$$dy \sim \Delta y \sim f'(x) \Delta x$$

As a second approximation

$$dy \sim \Delta y \sim f'(x) \Delta x + \frac{f''(x)}{2!} (\Delta x)^2$$

This technique is often useful in root finding problems where an initial guess is made from which better estimates are determined.

A still better approximation can be obtained by taking more terms in the expansion.

COMPUTATION BY SERIES

You may wonder how 15 place tables of the trigonometric, logarithmic, and exponential functions were obtained. The answer is computations by series.

The Maclaurin expansion for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

If $x = 1$, this becomes

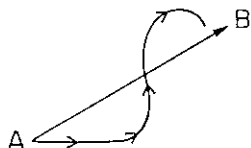
$$\begin{aligned} e &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!} + \dots \\ &= 1 + 1 + 0.5 + 0.166667 + 0.041667 \\ &\quad + 0.008333 + 0.001389 + 0.000198 + \dots \end{aligned}$$

Hence, approximately, $e = 2.718254$, which is correct to 4 decimals.

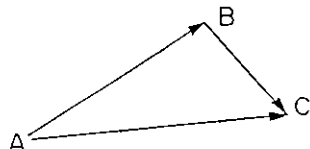
The gamma function can be evaluated using the series

$$\Gamma(n) = \int_0^1 \left(\log \frac{1}{x} \right) dx = \frac{1}{n} \sum_{m=1}^{\infty} \frac{\left(1 + \frac{1}{m} \right)^n}{1 + \frac{n}{m}}$$

Displacement is the position change of a particle. If a particle moves from position A to position B, we can represent its displacement by drawing a line from A to B; the direction of displacement can be shown by putting an arrowhead at B indicating that the displacement was from A to B. The path of the particle need not necessarily be a straight line from A to B; the arrow represents only the net effect of the motion, not the actual motion. A displacement is characterized by a length and a direction.



In a similar way, we can represent a subsequent displacement from B to C. The net effect of the two displacements will be the same as a displacement from A to C. AC is the sum or resultant of the displacements AB and BC. Notice this sum is not an algebraic sum and that a number alone cannot uniquely specify it.



Quantities that behave like displacements are called vectors. Vectors are quantities that have both magnitude and direction and combine according to certain rules of addition. Physical quantities which are vectors are force, velocity, acceleration, electric field strength, and magnetic induction.

VECTORS AND SCALARS

VECTORS

SCALARS

Quantities that can be completely specified by a single number and, therefore, have magnitude only are called scalars. Physical quantities which are scalars are mass, length, time, density, temperature, and energy. Scalars can be manipulated by the rules of ordinary algebra.

ADDITION OF VECTORS GEOMETRICAL METHOD

A vector representation on a diagram is an arrow. The arrow's length represents the magnitude of the vector, with the arrowhead giving the direction. The symbol "+" has a meaning for vectors different from ordinary algebra. The addition operations are different for vectors.

$$\vec{a} + \vec{b} = \vec{c}, \text{ as in the example above.}$$

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \quad (\text{Commutative law})$$

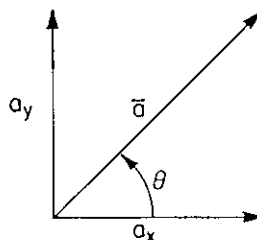
$$\vec{d} + (\vec{e} + \vec{f}) = (\vec{d} + \vec{e}) + \vec{f} \quad (\text{Associative law})$$

These laws show that the order or grouping of vectors is irrelevant when adding; the sum is the same. Subtraction can be included in vector algebra by defining the negative of a vector as another vector of equal magnitude and opposite direction. Then,

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}).$$

RESOLUTION OF VECTORS

The geometrical method of adding vectors is not very useful for vectors in three dimensions; often it is inconvenient for the two-dimensional case. Another way of adding vectors is the analytical method, involving the resolution of a vector into components with respect to a particular coordinate system.



The components a_x and a_y are readily found: $a_x = \vec{a} \cos \theta$ and $a_y = \vec{a} \sin \theta$ where θ is the angle that the vector $\vec{a}$ makes with the positive x-axis, measured counterclockwise from this axis. The components of a vector behave like scalar quantities because, in any particular coordinate system of a given reference frame, only a number, with an algebraic sign, is needed to specify them.

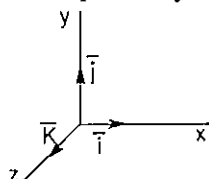
Once a vector is resolved into its components, the components themselves can be used to specify the vector. Instead of the polar form (magnitude and direction) the rectangular form (a_x and a_y) may be used. A vector is easily changed from one form to the other. To obtain a and θ from a_x and a_y we note from the figure on the previous page,

$$a = \sqrt{a_x^2 + a_y^2} \text{ and } \tan \theta = \frac{a_y}{a_x}.$$

The quadrant in which θ lies is determined from the sign of a_x and a_y .

When resolving a vector into components it is useful to introduce a vector of unit length in a given direction.

It is convenient to draw unit vectors along the particular coordinate axes chosen. In the rectangular coordinate system the special symbols $\bar{i}$, $\bar{j}$, and $\bar{k}$ are usually used for unit vectors in the positive x-, y-, and z-directions, respectively.



Vector $\bar{a}$ in the example above may be written in terms of its components and the unit vectors as

$$\bar{a} = \bar{i}a_x + \bar{j}a_y$$

Consider now the addition of vectors by the analytical method. Let $\bar{r}$ be the sum of the two vectors $\bar{a}$ and $\bar{b}$ lying in the x - y plane, so that $\bar{r} = \bar{a} + \bar{b}$. In a given coordinate system, two vectors such as $\bar{r}$ and $\bar{a} + \bar{b}$ can be equal only if their corresponding components are equal.

$$r_x = a_x + b_x$$

$$r_y = a_y + b_y$$

These two algebraic equations, taken together, are equivalent to the single vector relation

$$\bar{r} = \bar{a} + \bar{b} \quad r = \sqrt{r_x^2 + r_y^2} \quad \text{and } \tan \theta = \frac{r_y}{r_x}$$

POLAR RECTANGULAR

UNIT VECTOR

VECTOR ADDITION ANALYTICAL METHOD

MULTIPLICATION OF VECTORS

DOT PRODUCT

CROSS PRODUCT

The following is the analytic rule for adding vectors:

Resolve each vector into its components in a given coordinate system; the algebraic sum of the individual components along a particular axis is the component of the resultant vector along that same axis; the resultant vector can be constructed once its components are known. This method for adding vectors may be generalized to many vectors and to three dimensions.

The advantage of breaking vectors into components, rather than adding directly with the use of suitable trigonometric relations, is that one always deals with right triangles; and therefore calculations are simplified.

In the previous discussion it was assumed the vectors being added together were of like kind (velocity vectors added to velocity vectors). Like scalars, vectors of different kinds can be multiplied by one another to generate quantities of new physical dimensions. Because vectors have direction as well as magnitude, vector multiplication cannot follow exactly the same rules as the algebraic rules of scalar multiplication. There are new rules of multiplication for vectors.

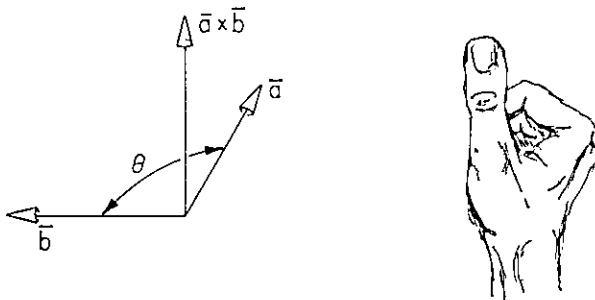
The multiplication of a vector by a scalar has a simple meaning: The product of a scalar K and a vector $\vec{a}$, written $K\vec{a}$, is defined as a new vector whose magnitude is K times the magnitude of $\vec{a}$. The new vector has the same direction as $\vec{a}$ if K is positive and the opposite direction if K is negative.

When multiplying a vector quantity by another vector quantity, it is necessary to distinguish between the scalar (or dot) product and the vector (or cross) product. The scalar product of two vectors $\vec{a}$ and $\vec{b}$, written as $\vec{a} \cdot \vec{b}$, is defined as $\vec{a} \cdot \vec{b} = a b \cos \theta$ where a is the magnitude of vector $\vec{a}$, $|b|$ is the magnitude of vector b , and $\cos \theta$ is the cosine of the angle θ between the two vectors. (There are always two different angles between a pair of vectors, depending on the sense of rotation. We always choose the smaller of the two in vector multiplication.)

Since a and b are scalars and $\cos \theta$ is a pure number, the scalar product of two vectors is a scalar. Several important physical quantities can be described as the scalar product of two vectors. Some of them are mechanical work, gravitational potential energy, electrical potential, electric power.

The vector cross product of two vectors $\vec{a}$ and $\vec{b}$ is written $\vec{a} \times \vec{b}$ and is another vector $\vec{c}$, where $\vec{c} = \vec{a} \times \vec{b}$. The magnitude of $\vec{c}$ is defined by $|c| = |a| |b| \sin \theta$, where θ is the smaller angle between $\vec{a}$ and $\vec{b}$.

The direction of $\vec{c}$, the vector product of $\vec{a}$ and $\vec{b}$, is defined to be perpendicular to the plane formed by $\vec{a}$ and $\vec{b}$; the vector $\vec{c}$ also passes through the intersection point of $\vec{a}$ and $\vec{b}$.



Imagine rotating a right-handed screw (whose axis is perpendicular to the plane formed by $\vec{a}$ and $\vec{b}$) from $\vec{a}$ to $\vec{b}$ through the angle θ between them. Then the direction of advance of the screw gives the direction of the vector product $\vec{a} \times \vec{b}$.

RIGHT HAND RULE

Another convenient way to obtain the direction of a vector product is the following. Imagine an axis perpendicular to the plane of $\vec{a}$ and $\vec{b}$ through their origin. Now wrap the fingers of the right hand around this axis and push the vector $\vec{a}$ into the vector $\vec{b}$ through the smaller angle between them with the fingertips, keeping the thumb erect; the direction of the erect thumb then gives the direction of the vector product $\vec{a} \times \vec{b}$ ($\vec{a}$ cross $\vec{b}$).

Notice that $\vec{b} \times \vec{a}$ is not the same vector and as $\vec{a} \times \vec{b}$, so that the order of factors in a vector product is important.

Examples of physical quantities that are vector products are torque, angular momentum, the force of a moving charge in a magnetic field, and the flow of electro-magnetic energy.

A number which is composed of a real and an imaginary part such as $a + bi$ is called a complex number ($i = \sqrt{-1}$). If a is zero, so that the number is of the form bi , the number is purely imaginary.

COMPLEX NUMBERS

It is also useful to note a few facts concerning powers of i

$$i = \sqrt{-1}$$

$$i^4 = i^2 \cdot i^2 = (-1) (-1) = +1$$

$$i^8 = i^4 \cdot i^4 = +1$$

Note that successive integral powers of i take on a repeating sequence of values $-1, -i, +1, i$, etc.

The fact that every power of i which is a multiple of 4 has the value 1, enables one to make such simplifications as:

$$i^{23} = (i^4)^5 \cdot i^3 = i^3 = -i.$$

Try program 09100-70007 on these examples.

FUNDAMENTAL OPERATIONS

Addition

$$\begin{array}{r} 2 + 4i \\ -5 + i \\ \hline -3 + 5i \end{array} \qquad \begin{array}{r} a + bi \\ c + di \\ \hline (a + c) + (b + d)i \end{array}$$

The sum of two complex numbers is in general a complex number whose real part consists of the algebraic sum of the real parts of the two numbers, and whose imaginary part is the algebraic sum of the imaginary parts.

Subtraction

$$\begin{array}{r} \text{minus} \quad 2 + 4i \\ -5 + 3i \\ \hline 7 + i \end{array} \qquad \begin{array}{r} a + bi \\ c + di \\ \hline (a - c) + (b - d)i \end{array}$$

Notice how the accumulators e and f can be used for these operations.

Multiplication

$$\begin{array}{r} 2 + 4i \\ -3 + 2i \\ \hline -6 - 12i \\ + 4i + 8i^2 \\ \hline -6 - 8i + 8i^2 = \\ -6 - 8i - 8 = \\ -14 - 8i \end{array} \qquad \begin{array}{r} a + bi \\ c + di \\ \hline ac + bci \\ + adi + bdi^2 \\ \hline (ac - bd) + (ad + bc)i \end{array}$$

Determine the keyboard operations for multiplication of complex numbers.

The operation is easily performed on the 9100A by placing the complex numbers in trigonometric form $z = x + iy = re^{i\theta}$.

Division

The division of complex numbers involves the process of rationalization of the denominator. You should review the idea of complex conjugates in connection with the following examples.

$$\frac{2+4i}{3-2i} = \frac{2+4i}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{6+16i+8i^2}{9-4i^2} = \frac{-2+16i}{13} = \frac{-2}{13} + \frac{16}{13}i$$

$$\begin{aligned} \frac{a+bi}{c+di} &= \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(ac+bd) + (bc-ad)i}{c^2-d^2i^2} = \frac{(ac+bd)}{c^2+d^2} + \frac{(bc-ad)i}{c^2+d^2} \\ &= Z = X + Yi \end{aligned}$$

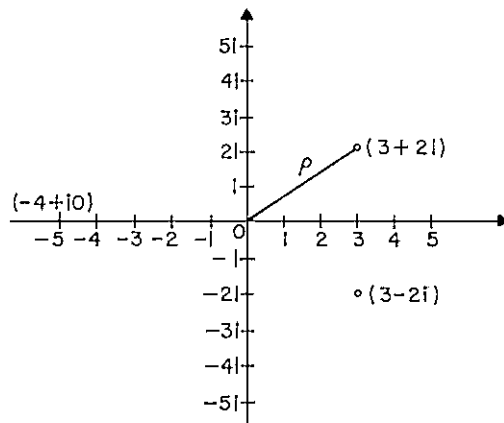
Determine the keyboard operations for division of complex numbers.

In polar form,

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1} y/x$$

The graphical interpretation of complex numbers is accomplished by using a set of rectangular axes, such as was done with the drawing of graphs of functions. The real numbers are laid off along the horizontal axis, called the real axis. The pure imaginary values are laid off along the vertical axis, called the axis of imaginaries. The unit 1 is used as a directed distance on the real axis; the unit i is used as a directed distance along the axis of imaginaries.

COMPLEX PLANE



The plane is called the complex plane. Complex numbers in general are points of the plane other than those along the X axis.

The figure shows the number $3 + 2i$, represented as a point 3 units over and 2 units up from the origin. If this point is joined to the origin, the length, represented by ρ in the figure is easily seen to be $\sqrt{3^2 + 2^2} = \sqrt{13}$

In general the distance of any complex number from the origin of the complex plane is given by

$$\rho = \sqrt{a^2 + b^2}$$

where the number is given as $a + bi$. ρ is called the absolute value of the complex number, and is often written

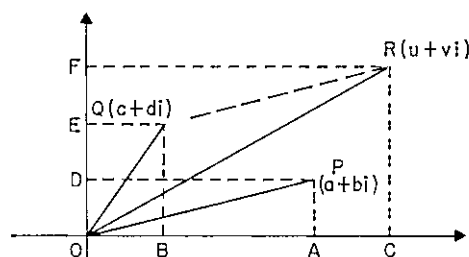
$$\rho = |a + bi| = \sqrt{a^2 + b^2}$$

Absolute Value

Graphical Operations

Note that all complex numbers which lie on a circle whose center is the origin will have the same absolute value, even though the values of a and b are different for each number. If two complex numbers are the same, they must represent the same point. Hence, if $a + bi$ and $c + di$ are equal, then $a = c$ and $b = d$.

In the figure below, let $P(a + bi)$ and $Q(c + di)$ represent two complex numbers. Then $R(u + vi)$ represents their sum. This follows from the sum of two complex numbers as given earlier and the fact that $OC = OA + OB$, and $OF = OD + OE$.



The figure OPRQ is a parallelogram. The distance OR is the diagonal of this parallelogram drawn from the origin. One interpretation of complex numbers is to consider them as represented by lines such as OP, OQ, OR. This is called the vector representation, and the vector (such as OP) represents both magnitude and direction. Such representation is useful in problems of physics which involve forces, velocities, electric current theory, etc.

The difference of two complex numbers is easily obtained from the figure if we assume that we wish to find $(u + vi) - (a + bi)$. This difference is given by $(c + di)$. It is obtained graphically by adding the negative of $a + bi$ (i. e., $-a - bi$) to the number $u + vi$.

ROOTS OF UNITY

Consider the problem of finding the cube roots of unity. That is, the numbers whose cubes are unity. Solve the equation $x^3 = 1$.

Write this as $x^3 - 1 = 0$

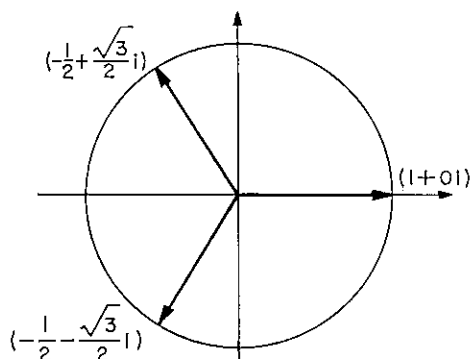
Factor $(x - 1)(x^2 + x + 1) = 0$.

This first factor yields a solution, $x = 1$.

The second factor by the quadratic formula gives

$$x = \frac{-1 + \sqrt{-3}}{2} \quad \text{or} \quad \frac{-1 - \sqrt{-3}}{2}$$

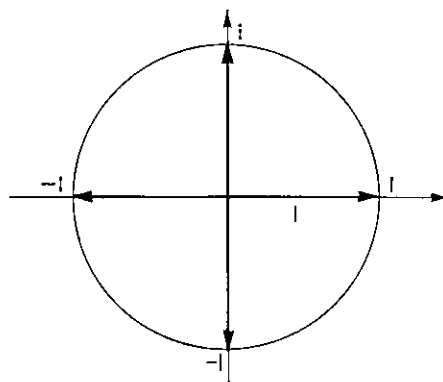
These are therefore the three cube roots of unity. Each of them has the same absolute value, namely 1. Hence, they all lie on a circle of radius 1.



The vectors representing these three complex numbers divide the unit circle into three equal parts. We can find the fourth roots of unity by solving $x^4 - 1 = 0$

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x - i)(x + i) = 0$$

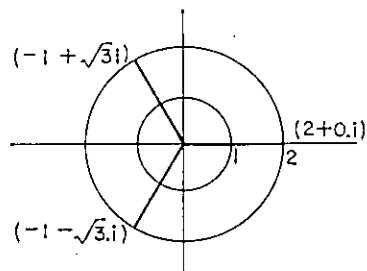
The roots are $x = 1, -1, i,$ and $-i$. The figure below shows these values and reveals the vectors representing these solutions divide the unit circle into four equal parts.



We solve the equation $x^3 - 8 = 0$

$$(x^3 - 8) = (x - 2)(x^2 + 2x + 4) = 0$$

The solutions given by these factors are $x = 2, -1 + \sqrt{3}i, -1 - \sqrt{3}i$. They are each just double the cube roots of unity. Hence, they each lie on a circle of radius two units.



CUBE ROOTS
OF EIGHT

RELATION BE-
TWEEN EXPO-
NENTIAL AND
TRIGONOMETRIC
FUNCTIONS

Notice when one has solved for the cube roots of unity, he has a method for finding the cube roots of any other number.

What are the cube roots of 17?

$$\sqrt[3]{17}, \sqrt[3]{17} \frac{-1 + \sqrt{3}i}{2}, \text{ and } \sqrt[3]{17} \frac{-1 - \sqrt{3}i}{2}$$

In general, the cube roots of any number A are equal to the real cube roots of A, multiplied by the three cube roots of unity.

Use the library program that solves for the roots of up to and including 5th degree polynomials on the previous examples.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!} + \dots.$$

It can be shown that these series converge for all values of x, real or complex. Indeed when $x = \alpha + i\beta$, these series will serve as definitions of $e^{\alpha+i\beta}$, $\sin(\alpha + i\beta)$, $\cos(\alpha + i\beta)$ respectively. For $x = i\theta$, a pure imaginary number, we have:

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots,$$

Since $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc. Multiplying the series for $\sin x$ by i and writing θ for x yields

$$i \sin \theta = i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \dots.$$

For $x = \theta$, the series for $\cos x$ becomes

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots.$$

Notice that the series for e^x can be obtained by adding the series for the two trig functions, i. e.,

$$e^{i\theta} = \cos \theta + i \sin \theta$$

EULER'S
IDENTITY

This is a remarkable relation and is generally known as Euler's Identity. It exhibits a very simple relation between $\sin \theta$, $\cos \theta$, and $e^{i\theta}$. Evidently $e^{i(-\theta)} = \cos(-\theta) + i \sin(-\theta)$

or
$$e^{-i\theta} = \cos \theta - i \sin \theta$$

Solving the two exponentials simultaneously for $\sin \theta$ and $\cos \theta$, we get

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

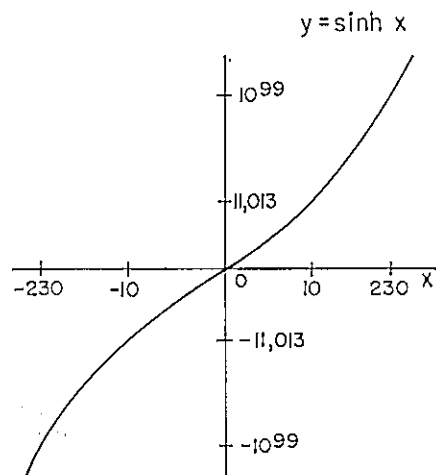
$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

These relations are very important and could be used as definitions of $\sin \theta$ and $\cos \theta$.

In many branches of applied mathematics the functions $\frac{e^{\theta} - e^{-\theta}}{2}$ and $\frac{e^{\theta} + e^{-\theta}}{2}$ frequently occur and are of definite importance. Note that the exponents are now real. Although these are just simple combinations of the exponential functions e^{θ} and $e^{-\theta}$, they are used so extensively that names have been given to them. They are called the "hyperbolic sine of the variable θ " and the "hyperbolic cosine of the variable θ " respectively. These are written " $\sinh \theta$ " and " $\cosh \theta$ ". That is, by definition

$$\sinh \theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

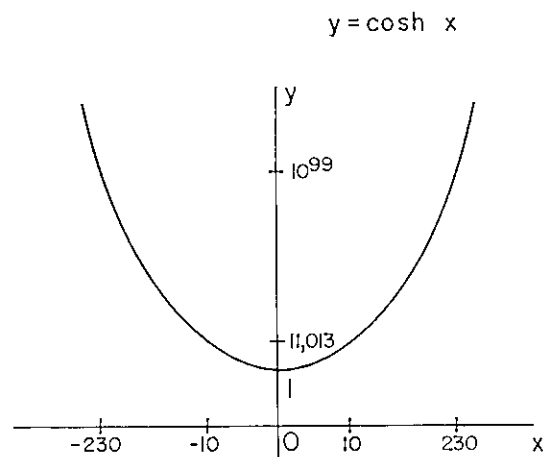
$$\cosh \theta = \frac{e^{\theta} + e^{-\theta}}{2}$$



HYPERBOLIC FUNCTIONS

ARC HYPER FUNCTIONS

FUNCTIONS OF COMPLEX VARIABLE



Direct hyperbolic functions are given in terms of exponentials; the inverse hyperbolic functions involve logarithms. For example, let

$$y = \sinh^{-1} x.$$

Then

$$\begin{aligned} x &= \sinh y \\ &= \frac{e^y - e^{-y}}{2}, \end{aligned}$$

which, on multiplying both sides by e^y , becomes a quadratic in e^y , namely,

$$e^{2y} - 2xe^y - 1 = 0.$$

Solving this we get

$$e^y = x \pm \sqrt{x^2 + 1},$$

where the minus sign must be discarded since e^y is always positive.

$$y = \sinh^{-1} x = \log_e (x + \sqrt{x^2 + 1}),$$

which holds for all values of x .

The following elementary functions are defined in the complex plane. The functions are all defined in terms of real functions and i , ($\sqrt{-1}$). Notice that when the imaginary part of the complex argument is zero, ($y = 0$), the complex definitions reduce to the real definitions. The notation $z = x + iy$ is used. x and y are real and $i = \sqrt{-1}$.

Exponential function:

$$e^z = e^x (\cos y + i \sin y)$$

y is the radian measure of the angle defining cos y and sin y.

Let y = 0

$$e^{x+i0} = e^x (\cos 0 + i \sin 0)$$

$$e^x = e^x$$

Trigonometric functions:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

A more useful form is $\cos z = \cos x \cosh y - i \sin x \sinh y$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

The more useful form is $\sin z = \sin x \cosh y + i \cos x \sinh y$

The following are definitions relating other complex functions

$$\tan z = \frac{\sin z}{\cos z}$$

$$\cot z = \frac{\cos z}{\sin z}$$

$$\sec z = \frac{1}{\sin z}$$

$$\csc z = \frac{1}{\cos z}$$

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

$$\tanh z = \frac{\sinh z}{\cosh z}$$

$$z = x + iy = re^{i\theta}$$

where

$$\theta = \tan^{-1} \frac{y}{x} \text{ and } r = \sqrt{x^2 + y^2}$$

$$\ln z = \ln r + i(\theta \pm 2n\pi)$$

$$n = 0, 1, 2, \dots \quad r > 0$$

TRIG
FUNCTIONS

HYPERBOLIC
FUNCTIONS

LOGARITHMIC
FUNCTIONS

COMPLEX EXPONENTS

$$z_3 = z_1^{z_2} = e^{z_2 \ln z_1}$$

$$\text{where } z_j = x_j + iy_j \quad n = 1, 2, 3$$

using the definitions of e^z and $\ln z$, the following computational formula is obtained.

$$\begin{aligned} z_3 &= e^{(x_2 + iy_2)(\ln \sqrt{x_1^2 + y_1^2} + i\theta_1)} \\ &= e^{x_2 \ln \sqrt{x_1^2 + y_1^2} - y_2 \theta_1} \left\{ \cos(y_2 \ln \sqrt{x_1^2 + y_1^2} + x_2 \theta_1) \right. \\ &\quad \left. + i \sin(y_2 \ln \sqrt{x_1^2 + y_1^2} + x_2 \theta_1) \right\} \end{aligned}$$

See program 09100-70007 for an example using this definition. Note the $\ln$ definition uses the principal value of θ , i.e., $-\pi < \theta \leq \pi$, so z_3 is uniquely specified.

APPLICATIONS

The purpose of the warning light on the calculator is to point out when an illegal operation is performed. Some illegal operations in the real plane are legal in the complex plane. An example is $\ln(-3)$. This operation is defined in the complex plane, i.e.

$$z = -3 + i0$$

$$\ln(-3) = \ln 3 + i(\theta \pm 2n\pi)$$

$$= \ln 3 \pm i(1 + 2n)\pi$$

$$n = 0, 1, 2, \dots$$

$$\theta = \tan^{-1} \frac{y}{x}$$

$$\theta = \tan^{-1} \frac{0}{-3}$$

$$= \tan^{-1}(-0)$$

$$= -\pi$$

Another example is $\sqrt{-10}$. The answer to this may be found using program 09100-70007.

In these examples the real functions on the calculator are used to evaluate the complex functions according to the definitions. It is convenient to output the complex results in the X and Y registers with the real part x in the X register and the imaginary part y in the Y register.

Use program 09100-70007 to evaluate the following. $2^3, 2^i, i^i, \sqrt{-10}$

Write a program to compute the $\cos z$. Apply your program to find $\cos(.5 + i1)$ Answer: $1.3542 - i 0.5634$

This section illustrates several methods of finding solutions of equations of the form $f(x) = 0$. That is, finding for a given function, $f(x)$, a value $x = x_r$ (x_r is real) such that $f(x_r) = 0$. In general, $f(x)$ will be a transcendental function.

ROOTS OF EQUATIONS

Before using any of these methods, it is necessary to know the approximate location of x_r . That is, one needs to know that the value $x = x_r$ lies in some interval $a < x_r < b$.

A rough approximation to the location of a root of $f(x) = 0$ is found by plotting the function $y = f(x)$

GRAPHICAL METHOD

Those values of x that yield $y = 0$ (graphically) are approximate values of the roots of $f(x) = 0$. Often it is difficult to plot $f(x)$ as a whole, so it is separated.

$$y_1 = f_1(x)$$

$$y_2 = f_2(x)$$

Where $f_1(x) + f_2(x) = f(x)$, values of x for which $f_1(x) = f_2(x)$ are approximations to the roots of $f(x) = 0$. Choose $f_1(x)$ and $f_2(x)$ so the plot is as simple as possible.

PLOTTING

The methods of plotting particular functions vary depending upon the functions involved. However, as a guide determine the intercepts on the coordinate axes; vertical and horizontal asymptotes and the behavior of the function as $x \rightarrow \pm \infty$.

Find an approximation to the real roots of

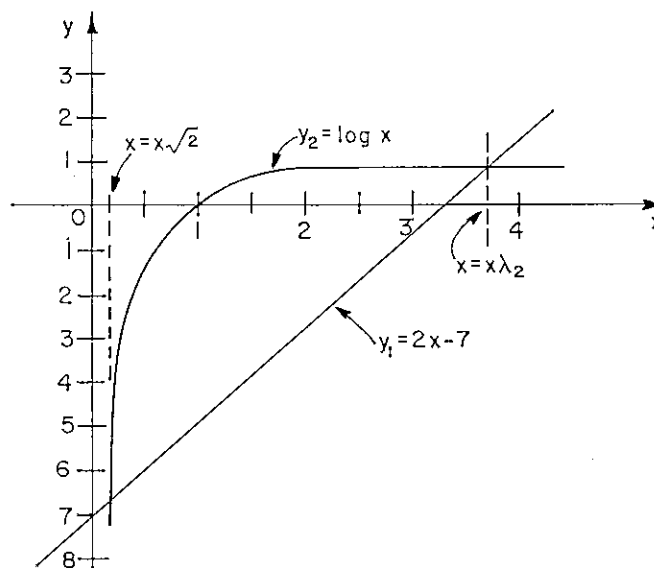
$$f(x) = 2x - \log x - 7 = 0$$

$$y_1 = f_1(x) = 2x - 7$$

and

$$y_2 = f_2(x) = + \log x$$

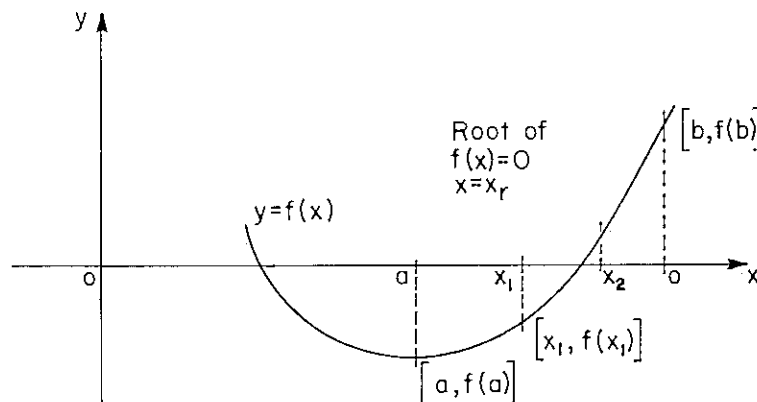
The plot of the functions appears in the figure below.



Observe the roots of $f(x) \approx 0$ located in the intervals $0 < x < 1$ and $3 < x < 4$.

METHOD OF BISECTION

This method is simple, yet it is very useful in obtaining approximate roots of $f(x) = 0$. Suppose one desires the roots of $f(x) = 0$ ($x = x_r$) that is known to lie in the interval $a < x_r < b$ (determined graphically).



The figure above shows $f(b) > 0$ and $f(a) < 0$. The method depends only on the sign of $f(x)$ for each estimate, x_i , ($i = 1, 2, 3, \dots$). The first estimate of the root is:

$$x_1 = a + (b - a) / 2$$

$$x_1 = \frac{a + b}{2}$$

The sign of $f(x)$ for $x = x_1$ is now negative. Thus the root of $f(x) = 0$ now lies in the interval $x_1 < x_r < b$. Proceeding in a like manner, the second approximation to the root, x_2 , is

$$x_2 = \frac{x_1 + b}{2}$$

Now $f(x_2) > 0$, thus $x_1 < x_r < x_2$, and so on until $x_1 = x_{i+1}$ to the desired number of significant figures as stated in the problem.

In general, the n^{th} approximation obtained by the above sequence of steps is in error by not more than

$$\frac{1}{2^n} (b - a).$$

Thus, an approximation to $x = x_r$ with an error less than 0.001 is

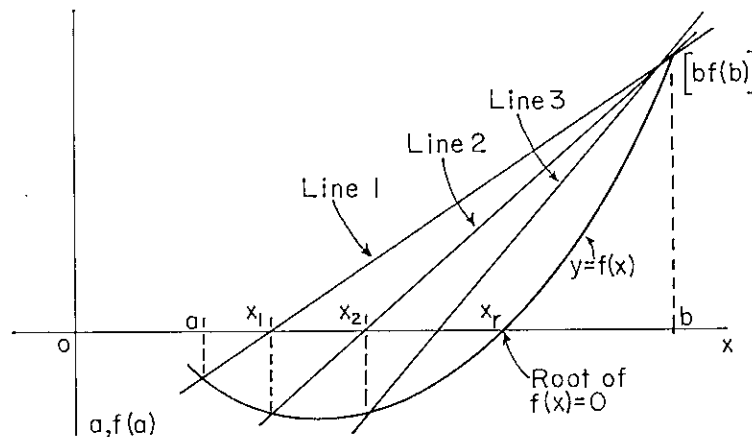
$$n = \frac{-\log(0.001) + \log(b - a)}{\log 2} \quad (\text{integral steps})$$

The method is very simple, but frequently too many steps are required to get a sufficiently accurate value. Also, the method will not work for multiple roots if the multiplicity is even.

The method of false position is one of the oldest root approximation methods known. It is a very simple method of improving an approximation to the roots of $f(x) = 0$ obtained by one of the methods previously discussed.

If $f(a)$ and $f(b)$ are of opposite sign, the plot of $f(x) = y$ crosses the x -axis in the interval $a < x < b$ at least once and a root $x = x_r$ of $f(x) = 0$ lies in this interval. The method uses a straight line approximation of the function $f(x)$ between these units. The point of intersection of the straight line and the x -axis approximates the root.

METHOD OF FALSE POSITION



From the geometry of the above figure

$$x_1 = a + \frac{f(a)}{f(a) - f(b)} (b - a)$$

Next determine $x - x_2$ which is a better approximation to $x = x_r$.

Replace x_1 by x_2 and a by x_1 to give:

$$x_2 = x_1 + \frac{f(x_1)}{f(x_1) - f(b)} (b - x_1)$$

Proceeding in a similar manner the n^{th} approximation to the root is:

$$x_n = x_{n-1} + \frac{f(x_{n-1})}{f(x_{n-1}) - f(b)} (b - x_{n-1})$$

In the figure, $f(b) > 0$ and $f(a) < 0$, note the necessary modifications in these equations if $f(b) < 0$ and $f(a) > 0$.

The process is continued until for some n , $x_{n-1} = x_n$ to a prescribed number of significant figures.

EXAMPLE

Determine the smallest positive roots of $f(x) = x^2 - \log x - 19 = 0$ correct to three significant figures. The root lies between $x = 3.16$ and 3.32 , since

$$f(3.16) = -9.5141 = f(a),$$

$$\text{and } f(3.32) = +0.5013 = f(b).$$

A better approximation, by the method of false position, is

$$x_1 = 3.16 + \frac{f(3.16)}{f(3.16) - f(3.32)} (3.32 - 3.16)$$

$$x_1 = 3.241, f(x_1) \approx -0.0066.$$

Continuing the method

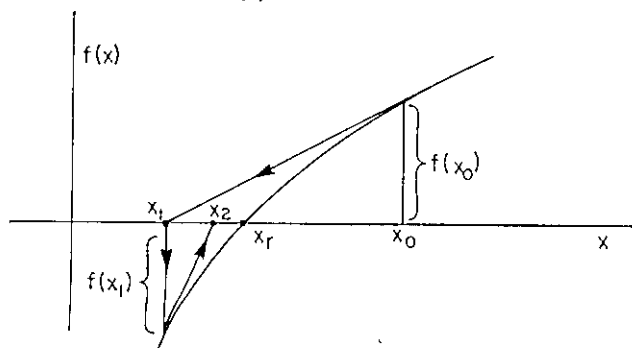
$$x_2 = 3.241 + \frac{f(3.241)}{f(3.241) - f(3.32)} (3.32 - 3.241)$$

Many of your customers want to solve algebraic equations of the third and higher orders as well as transcendental equations. Roots of these equations are frequently determined by an inefficient trial-and-error process. Although exact solutions are available for third and fourth-order equations, they are tedious to apply even with a calculator. The Newton-Raphson method, has the advantage of being applicable to all these types of equations.

As the examples show, convergence is rapid. In fact, the number of significant figures is approximately doubled at each step.

Newton's method, like the Method of Bisection and the Method of False Position, is an iterative one: An initial approximation is made for the solution and this approximation is improved step-by-step according to a definite procedure. It is a characteristic of iterative methods that minor errors at any step will usually be corrected in the following steps.

Let the equation whose real roots are to be determined be written as: $f(x) = 0$. A sketch of $f(x)$ is shown:



Let x_0 be a guess of x_r . x_0 may be obtained by graphical methods described earlier. Since $f(x)$ is an analytical expression we can differentiate $f(x)$ to get $f'(x)$. Substituting in x_0 we get $f'(x_0)$. Assuming x_1 is closer to our root x_r than x_0 , let's write an equation for x_1 in terms of x_0 and $f'(x_0)$.

$$f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

NEWTON'S METHOD

ITERATIONS

GENERAL FORM

Solving for x_1 :

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Writing a general expression for a new estimate (x_{i+1}) based on the last x_i

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad i = 0, 1, 2, \dots$$

This iterative method works very well on the 9100A. See Program #09100-72002 for examples.

SOLUTION STEPS

The steps in the application of Newton's method are:

1. Graph or tabulate $f(x)$ vs. x to find an approximate value x_0 of a real root.
2. Improve this starting value by calculating a corrected value x_1
3. Make successive corrections in the same manner
4. The solution is complete when the term $f(x_n)/f'(x_n)$ has no effect on the last decimal place to be retained in x_{n+1} .

Solve the algebraic equation, $x^3 - 4x = 8$, for a real root between $x = 2$ and $x = 3$ correct to five decimal places.

$$f(x) = x^3 - 4x - 8$$

$$f'(x) = 3x^2 - 4$$

Tabulate $f(x)$ vs. x to determine x_0 :

x	$f(x)$
2.000000	-8.000000
2.500000	-2.375000
3.000000	7.000000

The sign change tells us that there is a root between 2.5 and 3.0. Let $x_0 = 2.500000$.

It is convenient to arrange the succeeding steps of the solution in tabular form:

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	x_{n+1}
1	2.661017	0.198624	17.243034	0.011519	2.649498
2	2.749498	0.001059	17.059519	0.000062	2.649436
3	2.649436	0.000001	17.058533	0.000000	2.649436

You would probably display x_n , $f(x_n)$, and x_{n+1} in the 9100A's X, Y, Z registers.

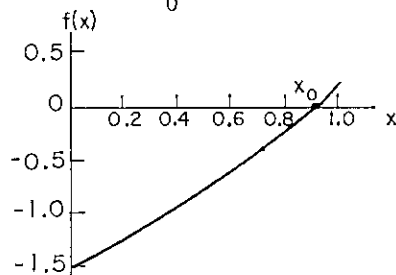
Since $x_3 = x_4$ (to the desired accuracy), the solution is $x = 2.64944$.

Solve the transcendental equation $x = 3/2 \cos x$, for a real root in the first quadrant correct to four decimal places.

$$f(x) = x - \frac{3}{2} \cos x$$

$$f'(x) = 1 + \frac{3}{2} \sin x$$

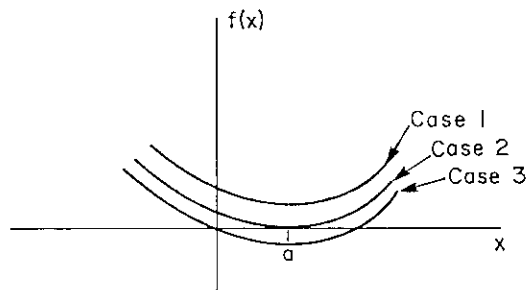
Graph $f(x)$ vs. x to determine x_0 :



The solution, to four decimal places, is $x = 0.9149$.

It is possible that $f'(x)$ may vanish in the neighborhood of a root. In this case, a different procedure must be used. First, solve for a value of x , $x = a$, such that $f'(a) = 0$. Then test for real roots of $f(x) = 0$ at $x = a$ using methods for determining relative maximum and minimum values of a function. There are three possibilities:

1. There is a double root at $x = a$ if $f(a) = 0$.
2. There is no root in the neighborhood of $x = a$ if $f(a) \neq 0$ and if both $f(a)$ and $f''(a)$ have the same sign.
3. There are two distinct roots in the neighborhood of $x = a$ if $f(a) \neq 0$ and if $f(a)$ and $f''(a)$ have unlike signs.



TRANSCENDENTAL EQUATIONS

EXCEPTIONAL CASES

COMPLEX ROOTS

Newton's method is valid for complex roots when the variable x is replaced by the complex variable, $z = x + yi$. Details of the procedure vary slightly since $f(z)$ is a function of two variables. An example will illustrate the initial steps in the solution.

Solve $z^3 - 1 = 0$, to four decimal places, for the complex root having a negative imaginary part. (This equation can, of course, be solved easily by factoring, and the roots are $z = 1$, $-0.5 + 0.8660i$, and $-0.5 - 0.8660i$.) To solve the equation by Newton's method, the expressions for $f(z)$ and $f'(z)$ must be determined:

$$f(z) = z^3 - 1$$

$$f'(z) = 3z^2$$

The starting value z is found by calculating $f(z)$ for various integral values of x and y . Let $z_0 = -1.00000 - 1.00000i$.

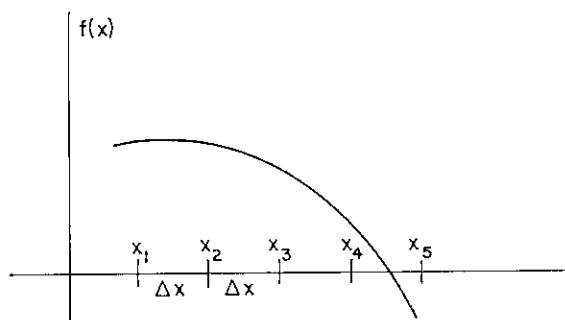
$$f(z_0) = 1.00000 - 2.00000i$$

$$f'(z_0) = 6.00000i$$

$$\begin{aligned} z_1 &= -1.00000 - 1.00000i - \frac{1.00000 - 2.00000i}{6.00000i} \\ &= -0.66667 - 0.83333i \end{aligned}$$

TRIAL AND ERROR

One may write a program for finding roots of equations of the form $f(x) = 0$ by trying various values of x at some increment Δx and watching for a sign change of $f(x)$.



$$x_{i+1} = x_i + \Delta x$$

$$i = 0, 1, \dots$$

When $f(x_{i+1})$ has a different sign than $f(x_i)$ the function has crossed the axis, provided the increment is small. See $f(x_4)$ and $f(x_5)$. Now one could reduce the increment, reverse direction, and repeat the process until x_{i+1} is sufficiently close to x_i .

Use program 09100-70014 to find 5 roots of $\tan x - x = 0$.

An equation involving one or more derivatives of a function is called a differential equation. Solution of a differential equation is an expression or formula for the dependent variable which is free of derivatives and when substituted in the differential equation reduces the expression to an identity.

DIFFERENTIAL EQUATIONS

An example of a differential equation is:

$$\frac{d^2 y}{dx^2} = -\sin x$$

A solution is $y = \sin x$, because

$$\frac{dy}{dx} = \cos x \quad \frac{d^2 y}{dx^2} = -\sin x$$

and substituting gives the identity

$$-\sin x = -\sin x$$

If the derivatives in the differential equation involve derivatives with respect to one independent variable, the equation is called an ordinary differential equation. If partial derivatives occur, the equation is called a partial differential equation. The order of a differential equation is the order of the highest derivative which appears in the equation.

Ordinary differential equation:

$$\frac{d^2 y}{dx^2} = -\sin x \quad (2\text{nd order})$$

Partial differential equation:

$$\frac{\partial^4 \mu}{\partial x^4} + 2 \frac{\partial^4 \mu}{\partial x^2 \partial y^2} + \frac{\partial^4 \mu}{\partial y^4} = 0 \quad (4\text{th order})$$

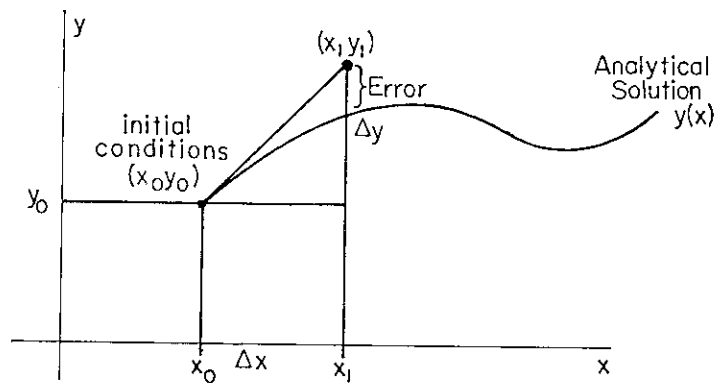
The following is a linear differential equation of the first order:

$$\frac{dy}{dx} = f(x, y)$$

With the initial conditions $y = y_0$ when $x = x_0$, a solution $y = g(x)$ that satisfies the differential equation and boundary conditions is desired. In an analytical method one seeks to obtain the functional relationship between the dependent variable y and the independent variable x . These may involve closed expressions or perhaps infinite series. In contrast,

a numerical solution expresses the functional relationship with tabulated data (x_1, y_1) or a graph over the desired range of x . An analytical solution is superior to a numerical solution but may be impossible to obtain. The superiority is due to the fact that for different boundary conditions the solutions only require a change of constants. In numerical solutions a complete recalculation is required.

Let $\frac{dy}{dx} = f(x, y)$ subject to the boundary conditions x_0, y_0 and solve for y as a function of x numerically. The solution must start at (x_0, y_0) as shown below:



An approximation of y_1 at $x_1 = x_0 + \Delta x$ is obtained by the following:

$$y_1 \cong \Delta y + y_0 \text{ where}$$

$$\Delta y = f(x_0, y_0) \Delta x$$

$$y_1 = y_0 + f(x_0, y_0) \Delta x$$

Everything on the right hand side of the equality is known.

$f(x_0, y_0)$ is the slope at (x_0, y_0)

$$\text{i.e. } \frac{dy}{dx}$$

The slope is used to predict y_1 . Also the smaller Δx , the more accurate will be y_1 .

Now knowing (x_1, y_1) the process to calculate (x_2, y_2) is similar.

Notice the following general form of the solution. The next point is based on the previous point.

$$y_{i+1} = y_i + \Delta y$$

There are numerous equations for y_{i+1} with different errors. The previous intuitive approach is crude but gives the basic idea. See the equation in 09100-70006 and notice the form.

The derivation of this Fourth-order Runge-Kutta formula is a tedious problem. The formula is:

RUNGE-KUTTA FORMULA

$$y_{i+1} = y_i + 1/6 (p + 2q + 2r + s)$$

$$(Note from y_{i+1} = y_i + \Delta y)$$

$$p = h \cdot f(x_i, y_i)$$

$$q = h \cdot f(x_i + 1/2h, y_i + 1/2p)$$

$$r = h \cdot f(x_i + 1/2h, y_i + 1/2q)$$

$$s = h \cdot f(x_i + h, y_i + r)$$

The values p, q, r and s are each estimates of the change in y on the interval x_i to x_{i+1} . p and s are estimates based on the beginning and end of the interval, respectively, while q and r are based on approximations to the slope at the mid-point of the interval. The Δy term added to y_i is a weighted average of several estimates. Note when f is independent of y the equation reduces to Simpson's One-Third Rule. Notice for each new point $f(x, y)$ must be evaluated four times.



CHAPTER VII
S T A T I S T I C S

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The probability of an event is defined as the relative frequency of that event. That is, the number of times the event occurred divided by the total number of times that the experiment was run. Thus, the probability of throwing a single die and getting a three is determined by throwing the die many, many times and dividing the number of times that a three comes up by the total number of throws. As you see, probability is really defined as the limit:

Probability of success=

$$\lim_{\text{trials} \rightarrow \infty} \frac{\text{number of successes}}{\text{number of trials}}$$

A ball is drawn at random from a box containing 6 red balls, 4 white balls and 5 blue balls. Determine the probability that it is (a) red, (b) white, (c) blue, (d) not red, (e) red or white.

Let R, W, and B denote the events of drawing a red ball, white ball and blue ball respectively. Then,

$$\begin{aligned} \text{(a) } \Pr\{R\} &= \frac{\text{ways of choosing a red ball}}{\text{total ways of choosing a ball}} \\ &= \frac{6}{6 + 4 + 5} = \frac{6}{15} = \frac{2}{5} \end{aligned}$$

$$\text{(b) } \Pr\{W\} = \frac{4}{6 + 4 + 5} = \frac{4}{15}$$

$$\text{(c) } \Pr\{B\} = \frac{5}{6 + 4 + 5} = \frac{5}{15} = \frac{1}{3}$$

$$\text{(d) } \Pr\{\bar{R}\} = 1 - \Pr\{R\} = 1 - \frac{2}{5} = \frac{3}{5}$$

$$\begin{aligned} \text{(e) } \Pr\{R, W\} &= \frac{\text{ways of choosing a red or white ball}}{\text{total ways of choosing a ball}} \\ &= \frac{6 + 4}{6 + 4 + 5} = \frac{10}{15} = \frac{2}{3} \end{aligned}$$

Another method:

$$\Pr\{R, W\} = \Pr\{\bar{B}\} = 1 - \Pr\{B\} = 1 - \frac{1}{3} = \frac{2}{3}$$

Note that $\Pr\{R, W\} = \Pr\{R\} + \Pr\{W\}$, i.e. $\frac{2}{3} = \frac{2}{5} + \frac{4}{15}$. This is an illustration of the general rule $\Pr\{E_1, E_2\} = \Pr\{E_1\} + \Pr\{E_2\}$ which is true for mutually exclusive independent events E_1 and E_2 .

Now assume there are two balls drawn. Each ball is replaced after the draw. What is the probability of: (a) drawing two red balls (b) drawing a red ball and a white ball.

$$\text{(a) } P\{RR\} = P\{R\} \cdot P\{R\} = \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}$$

$$\text{(b) } P\{RW\} = P\{R\} \cdot P\{W\} + P\{W\} \cdot P\{R\} = 2P\{R\} \cdot P\{W\} = 2 \cdot \frac{2}{5} \cdot \frac{4}{15} = \frac{16}{75}$$

INDEPENDENCE

Another important concept is independence. Two events are independent if the probability of the second one occurring is unchanged by the occurrence or non-occurrence of the first one.

RANDOM

Most of us have an intuitive feeling for what random means, but we aren't able to precisely define it. Random applies to the process by which a sample is chosen, not to the sample itself. Given five playing cards one cannot make a statement about the randomness of that particular poker hand without knowing how the particular five cards were chosen. The term random applies not to the cards in the hand, but rather to the method by which the cards were chosen. A random process allows no statement about a specific observation, but, on the other hand, does allow a statement about the relative frequency of an observation in a great number of trials.

EXPECTED VALUE

The last concept which is needed is that of expected value. Basically it is the value of the outcome multiplied by the probability of obtaining that outcome. Thus, the expected value in the throw of a single die is the sum of the value on each face times the probability of obtaining that face on the throw. This can be calculated as:

$$(1 \times 1/6) + (2 \times 1/6) + (3 \times 1/6) + (4 \times 1/6) + (5 \times 1/6) + (6 \times 1/6) = 21/6$$

$$E(x) = \sum_i P_i K_i$$

If a man purchases a raffle ticket, he can win a first prize of \$5000 or a second prize of \$2000 with probabilities 0.001 and 0.003. What is a fair price to pay for the ticket?

$$\begin{aligned}\text{Expectation} &= (\$5000)(0.001) + (\$2000)(0.003) \\ &= \$5 + \$6 = \$11,\end{aligned}$$

which is a fair price to pay.

SUMMARIZING DATA

One of the most common statistical problems is to describe a vast amount of data. There are many ways of doing this depending upon on how much data reduction we wish. The most straight-forward description would be a complete list of each of the data points. This certainly would be exhaustive; there is no way to more completely describe the data. There are, however, some obvious draw-backs to this approach. There may be so much information that it can't be digested.

68	84	75	82	68	90	62	88	76	93
73	79	88	73	60	93	71	59	85	75
61	65	75	87	74	62	95	78	63	72
66	78	82	65	94	77	69	74	68	60
96	78	89	61	75	95	60	79	83	71
79	62	67	97	78	85	76	65	71	75
65	80	73	57	88	78	62	76	53	74
86	67	73	81	72	63	76	75	85	77

RAW DATA

The table below contains exactly the same data, but in a format which makes analysis much easier.

RANKED DATA

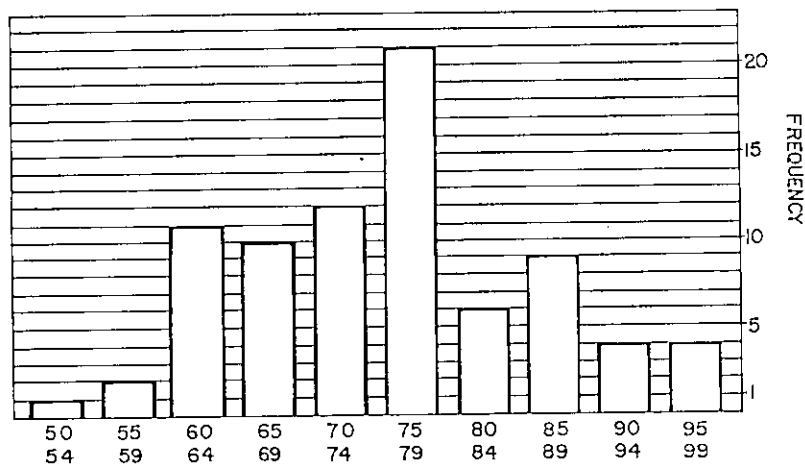
50-54	53
55-59	57, 59
60-64	60, 60, 60, 61, 61, 62, 62, 62, 62, 63, 63
65-69	65, 65, 65, 66, 67, 67, 68, 68, 68, 69
70-74	71, 71, 71, 72, 72, 73, 73, 73, 73, 74, 74, 74
75-79	75, 75, 75, 75, 75, 75, 75, 76, 76, 76, 76 77, 77, 78, 78, 78, 78, 79, 79, 79
80-84	80, 81, 82, 82, 83, 84
85-89	85, 85, 85, 86, 87, 88, 88, 88, 89
90-94	90, 93, 93, 94
95-99	95, 95, 96, 97

It is easily determined that:

HISTOGRAM

- (a) the highest grade is 97
- (b) the lowest grade is 53
- (c) the range is $97 - 53 = 44$
- (d) the five highest ranking students have grades 97, 96, 96, 96, 94
- (e) the five lowest ranking students have grades 53, 57, 59, 60, 60
- (f) the grade of the student ranking tenth highest is 88
- (g) the number of students receiving grades of 75 or high is 44
- (h) the number of students receiving grades below 85 is 63

We can also express these data in the form of a histogram.



FREQUENCY DISTRIBUTION

The data may be condensed even further:

<u>INTERVAL</u>	<u>FREQUENCY</u>	<u>RELATIVE FREQUENCY</u>
50-54	1	1/80
55-59	2	2/80
60-64	11	11/80
65-69	10	10/80
70-74	12	12/80
75-79	21	21/80
80-84	6	6/80
85-89	9	9/80
90-94	4	4/80
95-99	4	4/80
	<u>80</u>	

This table is much more concise, but it hides some of the information in the original data. Such an arrangement is called a frequency distribution since it gives the frequency with which each portion of the data occurs.

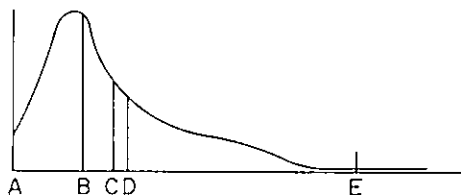
STATISTIC

A single number describing some feature of a set of data is called a "statistic". It is a more compact description than the frequency distribution. A statistic focuses attention more sharply than a frequency distribution on some feature of the data which it measures, but ignores more of the complexity of the data.

AVERAGES

The statistics most commonly requested by your customers will be "measures of location", commonly called "averages". They tell where a distribution is located, or centered, on the scale of conceivable values, more or less independently of other characteristics, such as its dispersion or shape.

As examples of measures of location, look at this drawing



The location of this distribution could be described by any of the following:

1. Its minimum, A
2. Its mode, B, the value of the variable at which the greatest concentration of observations occurs.
3. Its median, C, the value of the variable which exceeds half of the observations and is exceeded by half.
4. Its mean, D, the measure that the layman ordinarily refers to by the term "average".
5. Its maximum, E.

MODE

MEDIAN

Many other measures could be constructed.

If the curve had a rigid and known shape and size, any measure of location would be as useful as another, since from any one the others could be deduced. Fortunately, (because the absence of this condition sells calculators) this is rarely the case.

The arithmetic mean:

MEAN

$$\bar{X} = \frac{\sum_{i=0}^n x_i}{n}$$

is the most common statistic and is what most people are referring to when they say "average". There are also weighted means

$$\bar{X} = \frac{\sum_{i=0}^n W_i X_i}{\sum_{i=0}^n W_i}$$

Next to measures of location, such as the mean, the most common and useful statistics are measures of a frequency distribution's dispersion. The dispersion of a distribution, its spread or scatter, is sometimes described by measures such as the range, which is the difference between the largest and smallest observations. More commonly, dispersion is described by various averages of the deviations of the individual observations from the mean. Especially important is the root mean square deviation (that is, the square root of the mean of their squares), which is called the standard deviation.

DISPERSION

Take the deviation of the i th observation, x_i , from the mean, $\bar{x}$, of all observations, sum the squared deviations for all n observations, and divide by $n-1$:

$$s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}$$

VARIANCE

STANDARD DEVIATION

In computing this mean, one less than the number of cases (that is, $n - 1$) is used as the divisor, rather than the number of cases (n), because this makes the measure more useful for statistical inference, which is discussed later, and just as useful for statistical description, which is being reviewed now. This statistic, s^2 , is called the variance; the variance is of little importance as a descriptive statistic, but it is important in statistical inference and in combining samples.

The standard deviation, s , is then

$$s = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}}$$

The standard deviation has the least intuitive appeal or obvious interpretation of any statistical measure so important and so widely used. The reasons for its usefulness relate both to its practical merits for describing variability and to its role in statistical inference, that is, drawing conclusions about populations from samples.

Even though frequency distributions arise empirically in the analysis of experimental data, it is also possible to theoretically derive frequency distribution functions using the laws of probability. Where in the reduction of empirical data the term relative frequency was used, now the term probability is substituted.

RANDOM VARIABLE

A random variable is a number, associated with the outcome of a chance experiment, i. e., the number of heads obtained in repeated flips of a coin.

For now limit yourself to functions defined with discrete values. That is, the set of values such a random variable can assume is finite. The continuous case is treated later.

If x_0 is a possible value of a discrete random variable, the probability that the variable will assume this value will be written $P(x_0)$.*

It is customary to refer to the probabilities $f(x)$, specified for all values within the range of x , as values of the probability distribution or the probability function of x . How does one derive a probability function?

*It is common practice to write the probability that the random variable X assumes the value x as $P(X = x)$. This notation fails to distinguish between the random variable, a function, and the value of a random variable, the corresponding dependent variable.

BINOMIAL DISTRIBUTION

The most common discrete probability distribution is the binomial distribution; it applies to situations often referred to as "repeated trials." If we want to know the probability that 12 flips of a balanced coin will yield 5 heads and 7 tails, or that 3 of 10 persons will respond to a given mail questionnaire, we are interested in the probability of getting a certain number of "successes" in a given number of "trials".**

If a box contains red and black balls and a ball is drawn, there are only two possible outcomes: "red" and "black". We give the outcome "black" the value 1 with probability p . Then the value assigned to "red" is 0, with probability $q=1-p$. The model is summarized by the table

	x	$P(x)$
RED	0	$q = 1 - p$
BLACK	1	p

Suppose now that a random sample of n balls is drawn from such a box. Denote the result by $(x_1, x_2, \dots, x_n)$, where x_i is the value 0 or 1.

Consider the new random variable

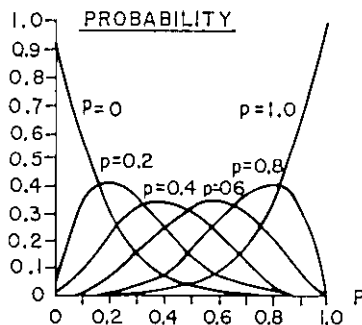
$$X = \sum_{i=1}^n x_i$$

What is X ? It is simply the number of 1's in our sample of 1's and 0's.

Your customer will want to know the frequency distribution of X , that is, the probability that X is some value y for any n and any y . The answer is called the binomial distribution.

$$P(X = y) = {}_n C_y p^y q^{n-y}$$

The numbers n and p are called parameters of the distribution. Thinking of the collection of all binomial distributions, it is the value of the parameters n and p , which describes a single distribution.



**This usage of "success", "failure", and "trials" is a holdover from the days when probability theory was considered only in connection with games of chance. Thus, the term "success" is not meant to imply that the outcome must necessarily be advantageous or desirable.

DERIVATION

One way to get $X = y$ is for $x_1 = 1, x_2 = 1, \dots, x_x = 1, x_{x+1} = 0, x_{x+2} = 0, \dots$, and $x_n = 0$. But because of the randomness of the sample, the probability of this is very small. The probability of picking a black ball for the first x tries is p^x . The probability of picking a red ball for tries x to n is q^{n-x} . Thus the probability of the above occurrence is $p^x q^{n-x}$.

Any other sets of draws for which $X = y$ will have x 1's among them and $n-x$ 0's. How many sets of draws of 0's and 1's make $X = y$? Or how many ways can we choose the x places to put the 1's? The answer to this is where

$${}_n C_y = \frac{n!}{y! (n-y)!}$$

Thus, there are ${}_n C_y$ arrangements of 0's and 1's which yield $X = y$.

But, the probability of a particular one of these arrangements being the outcome is $p^y q^{n-y}$. Hence, $P(X = y) = {}_n C_y p^y q^{n-y}$.

EVALUATION

For large values of n and y the numerical evaluation of ${}_n C_y [p^y q^{n-y}]$ becomes tedious. Large tables of this distribution for various values of n, y , and p have been published, but we have a better way--the 9100A program library contains a program to calculate ${}_n C_y$, and the $\ln x$ and e^x keys make the powers a cinch.

EXAMPLE

Here is an example. In a certain town 60 percent of the adult males are homeowners, what is the probability that a random sample of 10 adult males will contain 6 homeowners? Take the probability to be .60 that, when a single adult male is chosen, he will be a homeowner.

In this case the sample size, $n = 10$. The probability $p = 0.60$. Therefore: $q = 1 - p = .40$. The value of X is 6. Thus,

$$\begin{aligned} P(X = y) &= {}_{10} C_6 (.6)^6 (.4)^4 \\ &= \frac{10!}{6! 4!} (.6)^6 (.4)^4 = 0.251 \end{aligned}$$

Thus, the chances are about 1 in 4 that the sample of 10 will contain 6 homeowners.

What is the probability that the same sample will contain at least 6 homeowners?

Ans: .543

The Poisson distribution is a limiting form of the binomial distribution when $p \rightarrow 0$, $n \rightarrow \infty$, and np remains constant. It can be used to approximate the binomial distribution when $\lambda = np < 5$ and $n \geq 50$.

$$P(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{for } x = 0, 1, 2, \dots$$

The notation $(x; \lambda)$ indicates that it depends on one parameter, λ . The Poisson distribution can also be derived without reference to the binomial distribution, and it has many applications in its own rights; in other words, it has many applications where the binomial distribution does not apply.

When dealing with problems containing more than a countable number of outcomes, that is, in the continuous case, it is necessary to express probability measures with integrals. The probability that a random variable x assumes a value less than or equal to x_1 is

$$\int_{-\infty}^{x_1} f(x) \, dx$$

where $f(x)$ is the distribution function of x . Also

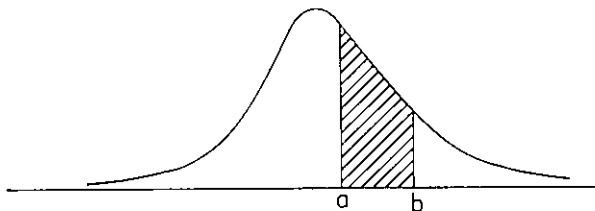
$$\int_{-\infty}^{+\infty} f(x) \, dx = 1$$

that is, something must happen!

The probability that it assumes a value somewhere in the interval from a to b is equal to

$$\int_a^b f(x) \, dx$$

You will often see this stated without the integrals by means of the figure below. The probability that the random variable x assumes a value on the interval from a to b is given by the shaded area, the area under the curve from a to b .



POISSON DISTRIBUTION

CONTINUOUS FREQUENCY DISTRIBUTIONS

DENSITY FUNCTION

Borrowing from the language of physics, $f(x)$ is known as the probability density or as the value probability density function at x .

Let's look at three continuous probability densities which are widely used; the exponential distribution, the gamma distribution, and the normal distribution. It would be more appropriate to refer to these probability densities as the uniform density, the exponential density, and so forth, but the practice of referring to them as distributions is deeply entrenched in the language of statistics.

EXPONENTIAL DISTRIBUTION

The exponential distribution has the form

$$f(x) = \frac{1}{\theta} e^{-(x/\theta)} \quad x > 0$$

with $\theta > 0$. The integral of this probability density, 0 to ∞ is equal to 1, i. e., something must happen sometime.

EXAMPLE

To illustrate how an exponential distribution might arise in actual practice, let us calculate the probability that a random variable having the exponential distribution does not assume a value less than or equal to x , namely

$$F(x) = \int_x^{\infty} \frac{1}{\theta} e^{-(x/\theta)} dx = e^{-(x/\theta)}$$

Now we would like to show that $e^{-(x/\theta)}$ is the probability that a piece of electronic equipment will not fail prior to time x if the probability of failure during any small time interval Δx is $(1/\theta) \Delta x$ independent of x . Dividing the interval from 0 to x into n equal parts $\Delta x = x/n$, the probability that the piece of equipment will not fail during one of these intervals is $1 - (1/\theta) \Delta x$ provided that Δx is very small. For large n , the probability that the piece of equipment will not fail during the time interval from 0 to x is, thus, $(1 - [1/\theta] \Delta x)^n$, which can be written

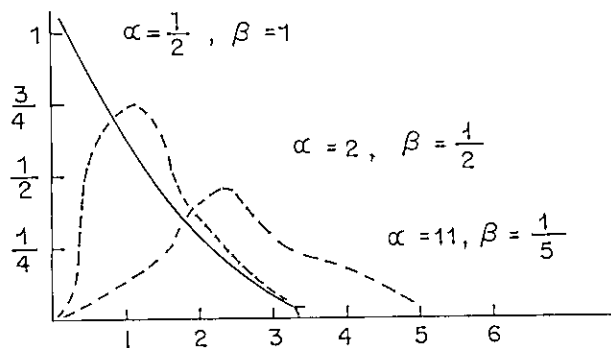
$$\left(1 - \frac{1}{\theta} \Delta x\right)^{(x/\Delta x)} = \left[1 - \frac{\Delta x}{\theta}\right]^{- (x/\Delta x)} = e^{-(x/\theta)}$$

If we now let Δx approach 0, the expression inside the square brackets approaches e , and the probability that the piece of equipment does not fail prior to the time x approaches $e^{-(x/\theta)}$. The exponential distribution, thus, provides the appropriate model for calculating probabilities concerning the "lifetime" of the given piece of equipment.

Another probability density which plays an important role in statistics is the gamma distribution

$$f(x) = \frac{x^{\alpha-1} e^{-(x/\beta)}}{\beta^{\alpha} \Gamma(\alpha)} \quad x > 0$$

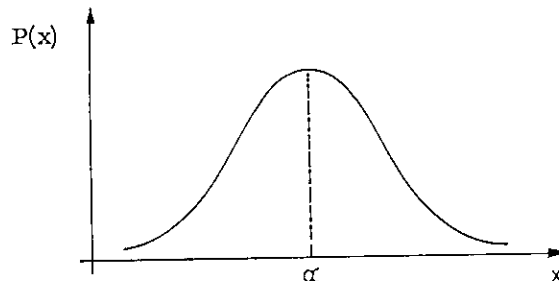
where $\alpha > 0$, $\beta > 0$ and $\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy$ for $\alpha > 0$. If α is an integer, n , then $\Gamma(\alpha) = \Gamma(n) = (n-1)!$. Ordinarily $\Gamma(\alpha)$ must be obtained from a table of values of the gamma function. You, however, are in a position to offer your customer a 9100A program. The graphs of several gamma distributions are shown below:



Notice that the parameters α and β are merely shaping parameters. They allow one to vary the shape of the curve to best fit the data.

Special cases of the gamma distribution play an important role in the theory of statistics. For $\alpha = 1$ the exponential distribution is obtained and for $\alpha = n/2$ and $\beta = 2$ the Chi-Square distribution is obtained.

The normal distribution is in many respects the cornerstone of modern statistical theory. It was studied first in the eighteenth century when scientists observed an astonishing degree of regularity in errors of measurement. They found that the error patterns observed were closely approximated by a continuous curve which they referred to as the "normal curve of errors".



GAMMA DISTRIBUTION

NORMAL DISTRIBUTION

The equation of the normal distribution, whose graph is shown on the previous page, is

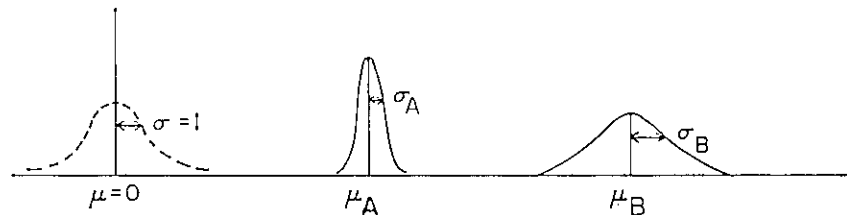
$$N(x; \alpha, \beta) = \frac{e^{-\frac{1}{2} \left(\frac{x-\alpha}{\beta} \right)^2}}{\beta \sqrt{2\pi}} \quad \text{for } -\infty < x < \infty$$

$$\text{let } \alpha = \mu \text{ and } \beta = \sigma$$

where α and β are real constants, $\beta > 0$. The normal distribution has a mean, μ and a standard deviation, σ .

SHAPING

By choosing appropriate values for the shaping parameters, μ and σ , the resulting normal distributions describe a great many populations.



This brings up a problem, however. The many density functions require that each one be individually integrated to find the probabilities that we are really interested in. This difficulty is eliminated, however, by transformation of variables,

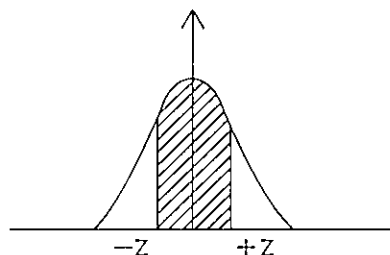
$$z = \frac{x-\mu}{\sigma}$$

The numerator moves the density function from its individual mean and centers it about the origin. The denominator adjusts the spread. The resulting standard normal density function:

$$N = \int_{-z}^{+z} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz$$

STANDARD GAUSSIAN DENSITY FUNCTION

has been extensively tabulated; your library has a program to evaluate its integral. Other common density functions don't have this convenient property; they must be individually integrated.



Several characteristics of the normal curve are worth remembering. The curve has inflection points at $z = \pm 1$. The area between $-1 < z < 1 = 0.67$; the area $-2 < z < 2$ is 0.987; the area $-3 < z < 3$ is 0.999. Thus, in a normally distributed population 2/3 of all observations are within one standard deviation of the mean.

Statistical inference is the process of reaching conclusions or making decisions on the basis of information contained in samples. For example, a statistical inference is concluding the "true" average lifetime of a certain kind of television tube is 5000 hours, or deciding (on the basis of appropriate experiments) that one variety of corn has a higher yield than another. Assuming the lifetime of the television tubes are random variables having exponential distributions with the parameter θ , the "true" average lifetime we are trying to estimate is, in fact, the parameter θ . Similarly, if the measurements of the yield of the two varieties of corn are assumed random samples from populations with the means μ_1 and μ_2 , the decision to be made is whether $\mu_1 > \mu_2$, or $\mu_1 < \mu_2$.

Traditionally, problems of statistical inference have been classified into problems of estimation and problems of testing hypotheses. As in the TV tube example, problems of estimation require that we provide values for unknown parameters of distributions; hypothesis testing, the corn example, deals with reaching conclusions or decisions concerning assumed values of the parameters.

A point estimate of the parameter in question is gotten by using a sample mean to estimate the mean of a population; a sample proportion to estimate the probability of success on an individual trial; or a sample variance to estimate the variance of a population.

Since point estimates based on samples don't usually give the values of the parameters they are supposed to estimate exactly, it is often desirable to give an interval rather than a single number. Then one can assert with a certain probability (or degree of confidence) that such an interval contains the parameter it is intended to estimate.

STATISTICAL INFERENCE

POINT ESTIMATION

INTERVAL ESTIMATION

PROPERTIES OF POINT ESTIMATORS

UNBIASED ESTIMATORS

For instance, when estimating the average I. Q. of all college students one might arrive at a point estimate of 117, or an interval estimate to the effect that the interval from 113 to 121 contains the "true" average I. Q. of all college students in the United States.

If the mean of a sample is used to estimate the mean of a population, the random variable $\bar{x}$ is referred to as an estimator of μ and the actual value which is obtained, say, $\bar{x} = 15.3$ is referred to as an estimate of μ . The statistic which used to estimate a parameter is referred to as the corresponding random variable.

When estimating the variance of a population, the sample variance, s , will not always equal the population variance, σ^2 exactly. However, it would be reassuring to know that this will happen at least on the average, namely, that $E(s^2) = \sigma^2$ (the expected value of the variance). Similarly, if one must decide whether to estimate the mean of a population with the sample mean or the sample median, it is important to know whether $\bar{x}$, the sample mean, or $\bar{X}$, the sample median, is more likely to yield a value that is actually close to the mean of the population.

Statistical properties of estimators are used to decide which estimator is most appropriate in a given situation. That is, which is more apt to give a desired result, which exposes one to the fewest risks, which gives the most information at the lowest cost, and so forth.

When estimating the parameter of a population, it is desirable that repeated experiments give values for the estimator that on the average equal the parameter for which they are an estimate. In other words, it is desirable that the expected value of an estimator equals the parameter it is supposed to estimate, in which case the estimator is unbiased. If the expected value of an estimator is not equal to the parameter it is intended to estimate it is biased.

Using this definition the sample mean, $\bar{x}$, is an unbiased estimator of σ since $E(\bar{x}) = \sigma$. Similarly, it is easy to show that x/n , where x is the number of successes in n trials, is an unbiased estimator of the parameter p (the probability of a success for one trial) of a binomial population.

When the sample variance, s^2 , was defined it was divided by $n-1$ rather than n . This made it an unbiased estimator of σ^2 for random samples from infinite populations. To prove:

$$\begin{aligned} E(s^2) &= E\left[\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2\right] \\ &= \frac{1}{n-1} E\left[\sum_{i=1}^n \{(x_i - \mu) - (\bar{x} - \mu)\}^2\right] \\ &= \frac{1}{n-1} \left\{ \sum_{i=1}^n E[(x_i - \mu)^2] - nE[(\bar{x} - \mu)^2] \right\} \end{aligned}$$

Using the fact that $E \left[(x_1 - \mu)^2 \right] = \sigma^2$ and $E \left[(\bar{x} - \mu)^2 \right] = \sigma^2/n$, it follows that

$$E(s^2) = \frac{1}{n-1} \left(\sum_{i=1}^n \sigma^2 - n \frac{\sigma^2}{n} \right) = \sigma^2$$

Note that s^2 is not an unbiased estimator of the variance of a finite population and that in either case s is not an unbiased estimator of σ .

If one has to select one of several unbiased estimators, it is customary to base the choice on their relative efficiency. If θ_1 and θ_2 are two unbiased estimators of θ , θ_1 is said to be relatively more efficient than θ_2 if $\text{var}(\theta_1)$ is less than $\text{var}(\theta_2)$. An actual measure of the relative efficiency of θ_1 with respect to θ_2 is given by the ratio

$$\frac{\text{var}(\theta_2)}{\text{var}(\theta_1)}$$

Thus, an unbiased estimator is particularly desirable if the variance of its distribution is small, that is, if the area under its density or distribution is concentrated closely about the mean.

Suppose one wants to estimate the mean of a normal population with a random sample of size n using either the median or the mean.

$\bar{x}$ is unbiased and $\text{var}(\bar{x}) = \sigma^2/n$. The variance of the median, which is also unbiased, is approximately $\pi\sigma^2/2n$. Hence, the sample mean is relatively more efficient than the sample median for estimating the mean of a normal population; the relative efficiency is $\pi/2$. The following is a somewhat more intuitive way of stating this result: since σ^2/n for $n = 64$ is approximately equal to $\pi\sigma^2/2n$ for $n = 100$, a sample mean based on 64 observations is "as reliable" as a sample median based on 100 observations.

Intuitively, it seems natural to estimate the mean of a population with the mean of a sample: to estimate the actual variance of, say, the strength of some kind of ceramic tile, with the variance of a sample. Although there are many instances where such "obvious" estimators are best, this is not always the case.

It would seem desirable to have some general methods yielding estimators which have the properties discussed in the preceding sections. One such method is the method of maximum likelihood. Another, the method of least squares, will be discussed later.

RELATIVE EFFICIENCY

EXAMPLE

METHODS OF POINT ESTIMATION

MAXIMUM LIKELIHOOD

In two papers published in the early 1920's, R. A. Fisher, proposed a general method of estimation, called the method of maximum likelihood. At the time he also demonstrated the advantages of this method by showing that for large n , the maximum likelihood estimators are most efficient.

In principle, the method of maximum likelihood selects that value of a parameter θ for which the probability (or probability density) of obtaining a given set of sample values is a maximum.

SAMPLES

You now know how to describe a population by its frequency distribution or parameters of its frequency distribution. This is fine if your customer can measure all members of the population. More often he can't, but instead he must estimate the population parameters from a sample.

A sample is often referred to as "the data" or "the observations": numbers that have been observed. The population or universe, on the other hand, is the totality of all possible observations of the same kind.

A single population can give rise to many different samples; thus, the population is stable but samples vary. A central problem is determining what inferences about the population can be drawn from the particular sample actually available in a practical problem.

REPEATED SAMPLES

A closed box contains an unknown number of red and green beads. Mix the beads by shaking the box, take a sample of 20, record the number of red beads, put the sample beads back into the box. Repeat 49 times, producing 50 samples in all.

This illustrates the kind of results expected if repeated samples are drawn with the same initial conditions. Since the beads from one sample were returned to the box before another sample was drawn, the population remained the same. It was as if one person studied the population of beads by drawing a single sample of 20 beads and counting the number of red beads, a second person studied the same population, and altogether 50 people studied it independently by exactly the same methods.

What is the proportion, P , of red beads? Always distinguish clearly between P , the proportion in the population, and p , the proportion in the sample. When P is unknown, one faces the statistical problem of basing our decision on the incomplete information about P given by p .

How does p (or any other sample statistic) vary from sample to sample? What appears in a single sample depends on chance, but the pattern of chance variation in repeated samples depends on the population.

SAMPLING DISTRIBUTIONS

These two facts correspond with the basic problem and the basic tool of statistical analysis. A sample is not a miniature replica of a population. So when decisions about a population are based on a sample it's necessary to make allowance for the role of chance in determining the characteristics of that particular sample. The basic tool is knowledge of the patterns of sample variability resulting from various populations. This gives the probability of getting the observed sample from any of the different populations with might conceivably have been its source.

The greater the variation among population items (that is, the less homogenous the population), the greater, other things equal, will be the variability among samples. In analyzing samples you will repeatedly fall back on the question, "Can the observed differences reasonably be explained by chance?" Only when the answer is "no" will the evidence imply assignable causes.

CHANCE?

In generalizing from the sample to the population, allowance must be made for the fact that the sample results are partly fortuitous. This allowance is made by considering the pattern of sampling variability, called the sampling distribution, of sample results.

Your 9100A program library deals with random samples from normal populations, namely, infinite populations of the form $N(x; \mu, \sigma^2)$.

In particular, one wants to calculate and use the distributions of the statistics $\bar{x}$, $s^2/\sqrt{n}$, $(\bar{x} - \mu)/s$, and s_1^2/s_2^2 , where s_1^2 and s_2^2 are the variances of independent random samples from two normal populations.

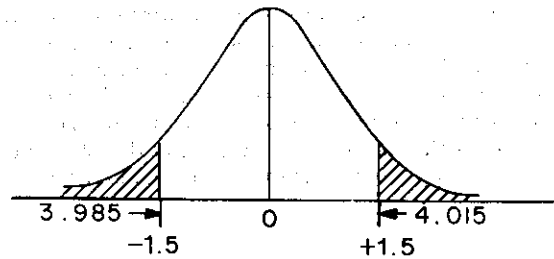
If $\bar{x}$ is the mean of a random sample of size n from the normal population $N(x; \mu, \sigma^2)$, then the sampling distribution of $\bar{x}$ is the normal distribution $N(\bar{x}; \mu, \sigma^2/n)$.

SAMPLING DISTRIBUTION OF MEAN

Many applications of this important fact exist. Let's take a typical example. A production process turns out thousands of compression springs each day. The spring lengths should average 4 inches, and the vast majority of springs should deviate less than 0.05 inches from this average. In other words, the process is under control if the length of any one of the springs (selected at random) is a value from a population having the distribution $N(x; 4, 0.0025)$. To check whether the process is under control, 25 springs are randomly selected from each day's production. The lengths of these samples will have the distribution $N(x; 4, 0.0001)$ and the process is considered under control

TEST MEAN

if the mean of their lengths falls within 1.5 standard deviations of the sampling distribution mean. What is the probability that a sample will fail to meet this criterion even though the process is under control?



Normal Distribution

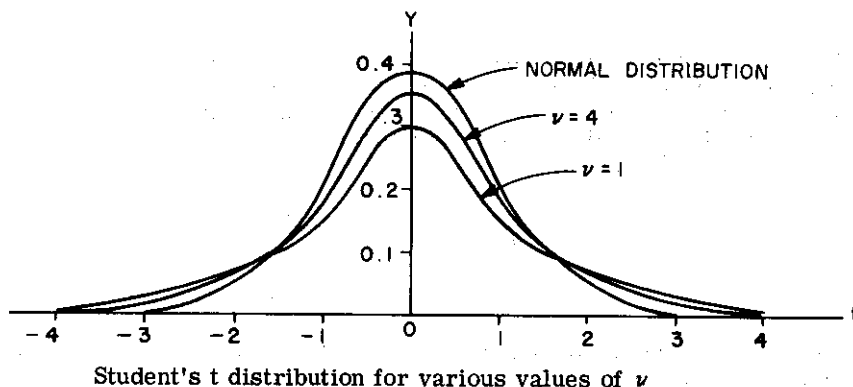
The answer to this is the ratio of the shaded area to the total area of the figure above, representing the probability that a random variable having the normal distribution $N(x; 4, 0.0001)$ assumes a value less than or equal to 3.985 or greater than or equal to 4.015. The slanted numbers label the normalized curve. Use your calculator to find the answer.

THE t DISTRIBUTION

In random samples from normal populations the sampling distribution of $(\bar{x} - \mu)\sqrt{n}/\sigma$ is the standard normal distribution. One difficulty in applying this is that in actual practice σ is usually unknown, which makes it necessary to replace it with an estimate. However, an exact sampling distribution, $f(t)$, for $(\bar{x} - \mu)\sqrt{n}/s$ for random samples from normal populations may be obtained.

$$f(t) = \frac{Y_0}{1 + \frac{t^2}{\nu}} \frac{(\nu + 1)/2}{2} \quad \text{for } -\infty < t < \infty$$

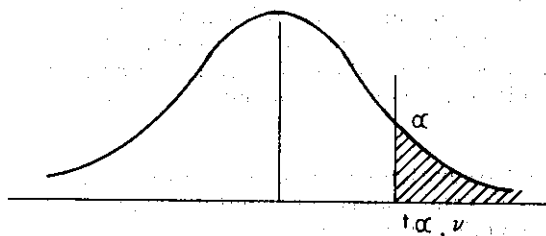
This distribution is called the t distribution (or the Student-t distribution) with ν (ν is the Greek letter nu; $\nu = n - 1$, n = sample size) degrees of freedom. Y_0 is a constant depending on ν such that the total area under the curve is one. The distribution was first obtained by W. S. Gosset, who at the time was employed by a well-known Irish brewery. Since this brewery did not permit the publication of research done by its staff, Gosset chose the pen name "Student"; hence, the name "Student-t distribution."



In view of its importance, the t distribution integral

$$\int_{1-\alpha}^{\infty} f(t) dt = \alpha$$

has been tabulated extensively, and there is a program in the 9100A library to evaluate this integral. Note that in view of the symmetry of the t distribution $t_{1-\alpha, \nu} = -t_{\alpha, \nu}$. When ν exceeds 30, the t distribution, or better the distribution of $t\sqrt{(\nu-2)/\nu}$ can be approximated with the standard normal distribution.



If $\bar{x}$ and s are the mean and the standard deviation of a random sample of size n from the normal population $N(x; \mu, \sigma^2)$, then $(\bar{x} - \mu)\sqrt{n}/s$ has the t distribution with $n-1$ degrees of freedom.

TEST MEAN WITH SMALL SAMPLE

To illustrate, let's check whether the mean of a given normal population is $\mu = 100$. A random sample of 16 had a mean of 108 and a variance of 9. We thus get

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{108 - 100}{3/\sqrt{16}} = 10.67$$

Since the probability of getting a value of t greater than 10.67 is 10^{-8} for 15 degrees of freedom; it would seem reasonable to reject the hypothesis that the sample came from a normal population whose mean is $\mu = 100$.

CHI SQUARE DISTRIBUTION

If s^2 is the variance of a random sample of size n from the normal population $N(x; \mu, \sigma^2)$, then ns^2/σ^2 has a chi-square distribution with $\nu = n - 1$ degrees of freedom. The chi-square distribution χ^2 is a special case of the gamma distribution with the shaping parameters $\alpha = \nu/2$ and $\beta = 2$.

$$f(x^2) = \int_0^{x^2} \frac{1}{2^{\nu/2} \Gamma(\nu/2)} x^{(\nu-2)/2} e^{-(x/2)} dx$$

for $x > 0$

The chi-square distribution has a mean and variance that are, respectively, ν and 2ν .

TEST VARIANCE

The chi-square distribution is used to develop interval estimates for standard deviations and to test hypotheses concerning variances. In the past the standard deviation of weights of certain 40.0 ounce packages filled by a machine was 0.25 ounces. A random sample of 20 packages showed a standard deviation of 0.32 ounces. Is the apparent increase in variability significant at the 0.05 level of significance?

The two hypotheses are:

$H_0: \sigma = 0.25$ ounces, and the observed result is due to chance

$H_1: \sigma > 0.25$ ounces, and the variability has increased

The value of χ^2 for the sample is $\chi^2 = ns^2/\sigma^2 = 32.8$. Hypothesis H_0 should be rejected if the probability of getting $\chi^2 = 32.8$ for 19 degrees of freedom is smaller than 5%. Using your program library, what is your decision? Should the manufacturer check his loading machine or not?

The chi-square distribution has a number of important mathematical properties. If x has the standard normal distribution, then x^2 has the chi-square distribution with 1 degree of freedom.

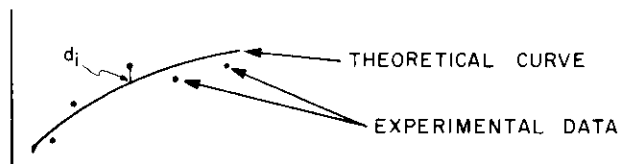
More generally, if $x_1, x_2, \dots, x_n$ are independent random variables having standard normal distributions, then

$$Y = \sum_{i=1}^n x_i^2$$

has the chi-square distribution with n degrees of freedom.

TEST GOODNESS OF FIT

This fact allows us to use the chi-square distribution to test the goodness of fit between experimental data and theoretical model.



The deviations, d_i , between the i th data point and the theoretical curve should be normally distributed if the data does, in fact, fit the curve. Therefore, $\sum (d_i)^2$ will have a chi-square distribution, and we can find the probability of discrepancies between expected (theoretical) and observed (experimental) points.

Occasionally the "theoretical" curve will be estimated from the data. This will reduce the number of independent deviations, d_i , by the number of curve parameters obtained from the data. For example, an infinite number of second order polynomials can be made to go thru two given points. Only one quadratic, however, will pass thru three given points. If there are four points thru which a quadratic equation has been fitted, three points are required to specify the fitted curve; only one point, the fourth, can be used as an independent check on the goodness of fit.

DEGREES
OF
FREEDOM

If x_1 and x_2 are independent random variables having chi-square distributions with ν_1 and ν_2 degrees of freedom; then the F distribution is given by

F - DISTRIBUTION

$$\text{Let } m = \nu_1 \\ n = \nu_2$$

$$f(x) = \int_0^F \frac{\Gamma \frac{m+n}{2} m^{\frac{m}{2}} n^{\frac{n}{2}} x^{\frac{m}{2}-2} (n+mx)^{-\frac{m+n}{2}}}{\Gamma \left(\frac{m}{2} \right) \Gamma \left(\frac{n}{2} \right)} dx$$

$$x > 0$$

One important application is comparing the variances of two normal populations; for example, in testing the hypothesis that $\sigma_1^2 = \sigma_2^2$. Recall

COMPARE
TWO
VARIANCES

$$\frac{(n_1 - 1) s_1^2}{\sigma_1^2} \quad \text{and} \quad \frac{(n_2 - 1) s_2^2}{\sigma_2^2}$$

TEST
MANY
MEANS

are independent random variables having chi-squares distributions with $n_1 - 1$ and $n_2 - 1$ degrees of freedom if the samples are independent, and random and also from normal populations. Assuming $\sigma_1^2 = \sigma_2^2$ then

$$\frac{\left(\frac{(n_1 - 1) S_1^2}{\sigma_1^2} \right) / (n_1 - 1)}{\left(\frac{(n_2 - 1) S_2^2}{\sigma_2^2} \right) / (n_2 - 1)} = \frac{S_1^2}{S_2^2} = F$$

has an F distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom. The probability of the hypothesis being true is the ratio of

$$\int_x^\alpha f(x) dx \text{ to } \int_0^\alpha f(x) dx.$$

You know how to compare means of two populations. How are mean values of many populations tested? The F distribution is the required tool. The technique is to consider the various samples first individually and then as one large sample.

Consider four samples each of size 6. Assume (for now) the population means are the same, these four samples can be regarded as four independent samples from the same normal distribution. Compute the sample means $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, and $\bar{x}_4$.

Compute the standard deviation of the four means and compare it with the standard deviation of the entire data. Actually, it is simpler to deal with variances, the squares of the standard deviations, and the corresponding relation

$$\sigma_{\bar{x}}^2 = \sigma^2 / n$$

The four means can be regarded as a sample from the population of means of all samples of size 6. The distribution of means will have a variance equal to the variance of the observations, divided by 6.

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{6}$$

$$\text{or } 6\sigma_{\bar{x}}^2 = \sigma^2$$

The analysis involves three stages: (1) Compute a sample estimate of $\sigma_{\bar{x}}^2$ treating the means as a sample. (2) Compute a sample estimate of σ^2 using the variation within the individual samples. (3) Finally,

compare these two sample estimates, allowing for some discrepancy due to sampling error. If the number obtained directly from the means is significantly bigger than the number obtained from the variation of the observations within samples, it may be concluded that something has been added to the ordinary sampling variation among the means--in other words, that the samples come from populations with different means.

Thus two independent estimates of σ^2 , s_A^2 , s_W^2 , are computed from these four samples. Under the hypothesis that the four samples are drawn from the same population, one would expect s_A^2 and s_W^2 to be the same, within the limits of the appropriate pattern of chance variation.

This method of testing the equality of a group of means is an elementary case of what is known as the analysis of variance. This elementary version assumes the variability is the same in all the populations being compared, whether or not the means are the same.

ANALYSIS OF VARIANCE

Had s_A^2 been larger than s_W^2 , we would have faced the question whether the excess was more than could be attributed to chance. This question is answered by computing the variance ratio, or F ratio, so named for Sir Ronald A. Fisher who introduced the method being discussed, and a wide range of related methods, in 1924:

$$F = \frac{s_A^2}{s_W^2}$$

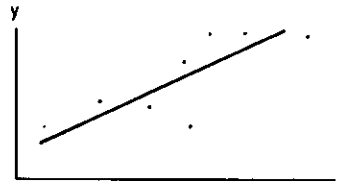
Statisticians are equipped with rather elaborate tables of the F distribution, in which they select a page corresponding with the significance level, a column corresponding with the number (k - 1) used in the denominator of s_A^2 , and a row corresponding with the numbers ($\sum n - k$) used in the denominator of s_W^2 . There they find the rejection level of F--the value which, if exceeded, leads to rejection of the hypothesis that the population means are equal.

It is often desirable to observe and measure the association between two or more statistical series. For instance, it may be desirable to know whether there is a relationship between changes in the cost of living and changes in wages; the amount of electrical current passed through a solution and the amount of substance deposited by electrochemical reaction; and many other similar associated (correlated) series.

LINEAR CORRELATION

SLOPE AND INTERCEPT

If two related series are plotted graphically with one variable placed on the X axis and the other on the Y axis, the result is known as a scatter diagram. The independent variable is placed on the X axis while the dependent variable is placed on the Y axis. The dependent variable is to be estimated from the other variable. If there is a definite relationship resulting from plotting the associated variables on a chart, the points will follow a definite line of movement or "path."



One would like to find the equation relating y to x and quantitatively measure the amount of correlation. Any straight line is completely specified by its intercept and slope, so the problem of fitting a linear regression line reduces to estimating the intercept and the slope.

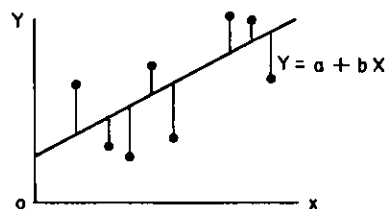
The problem can be looked at this way: There is a sample of n pairs (X, Y) of values from a population. Values of Y are independently normally distributed with means, M , that depend on X through the following equation:

$$M = A + BX.$$

The standard deviation of Y for a given X , denoted by $\sigma_{y,x}$, is assumed to be the same for all values of X , but is usually unknown. In effect, then, we have a series of separate populations for Y , one population for each value of X . These populations are assumed to be normally distributed. For any particular value of X the mean of the population of Y 's is $A + BX$. The problem is to make estimates, a and b , of the parameters A and B of the population, on the basis of a sample.

LEAST SQUARES

The principle of maximum likelihood leads to selecting a and b by what is called the method of least squares. This method selects a and b so the mean of the squared vertical deviations between the data and the fitted line is as small as possible.



A "vertical deviation" is the vertical distance from an observed point to the line, as depicted by the vertical segments. The deviation of each data point is squared and the line is placed so the sum of these squared deviations is a minimum. This could be done by trial and error, but that is not necessary. The least squares line can be found from convenient formulas for a and b.

The formula for the maximum likelihood or least-squares estimator of the slope is

$$b = \frac{\sum (X - \bar{X})(Y - \bar{Y})}{\sum (X - \bar{X})^2} .$$

SLOPE

This may be written in the following form, more convenient for computations:

$$b = \frac{\sum XY - \frac{(\sum X)(\sum Y)}{n}}{\sum X^2 - \frac{(\sum X)^2}{n}}$$

Once the estimated slope, b, is known, the estimated intercept, a, is easy to compute:

$$a = \bar{Y} - b\bar{X} = \frac{\sum Y}{n} - b \frac{\sum X}{n} .$$

INTERCEPT

The formula for any straight line will be:

$$Y_c = a + bX$$

LINE

where Y_c represents the computed or theoretical value for Y obtained by substituting the appropriate value in the formula.

If Y_{est} represents the calculated value of Y for given values of X, a measure of the scatter about the regression line of Y on X is supplied by the quantity

STANDARD
ERROR OF
ESTIMATE

$$s_{y.x} = \sqrt{\frac{\sum (Y - Y_{est})^2}{n - 2}}$$

which is called the standard error of estimate of Y on X.

An analogous standard error of estimate of X on Y is defined by

$$s_{x.y} = \sqrt{\frac{\sum (X - X_{est})^2}{n - 2}}$$

In general, $s_{y.x} = s_{x.y}$.

The standard error of estimate has properties analogous to those of the standard deviation. For example, if lines parallel to the regression line of Y on X are constructed at respective vertical distances $s_{y.x}$, $2s_{y.x}$, and $3s_{y.x}$ from it, one would find, if N is large enough, that there would be included between these lines about 68%, 95% and 99.7% respectively of the sample points. The points are normally distributed about the line.

In this estimate $n - 2$ is the number of degrees of freedom, instead of $n - 1$, because the deviations are based on two quantities (a and b) computed from the sample, not just one as in the simple standard deviation.

The regression equation $Y = a + bX$ is obtained on the basis of sample data. The following are two tests concerning the population from which the sample was drawn.

TEST SLOPE

To test the hypothesis that the regression coefficient is equal to some specified value, use the fact that the statistic

$$t = \frac{b - B}{s_{y.x} s_x} \sqrt{n - 2} = \frac{b - B}{\sqrt{1 - r^2}} \sqrt{n - 2}$$

has Student's t distribution with $n - 2$ degrees of freedom. This can also be used to find confidence intervals for population regression coefficients from sample values.

TEST PREDICTED VALUE

Let Y_o denote the predicted value of Y corresponding to $X = X_o$. Let Y_p denote the predicted value of Y corresponding to $X = X_o$ for the population. Then the statistic

$$t = \frac{Y_o - Y_p}{s_{y.x} \sqrt{1 + 1/n + (X_o - \bar{X})^2 / (ns_x^2)}}$$

has Student's t distribution with $n - 2$ degrees of freedom. From this, confidence limits for predicted population values can be found.

The total variation of Y is defined as $\sum (Y' - \bar{Y})^2$, i.e., the sum of the squared deviations between Y and the mean $\bar{Y}$. This can be written

$$\sum (Y - \bar{Y})^2 = \sum (Y - Y_{est})^2 + \sum (Y_{est} - \bar{Y})^2$$

The first term on the right is called the unexplained variation while the second term is called the explained variation, so called because the deviations $Y_{\text{est}} - \bar{Y}$ have a definite pattern while the deviations $Y - Y_{\text{est}}$ behave in a random or unpredictable manner. Similar results hold for the variable X .

The square root of the ratio of explained variation to total variation is called the correlation coefficient. If there is zero explained variation, i.e., the total variation is all unexplained, this ratio is zero. If there is zero unexplained variation, i.e., the total variation is all explained, the ratio is one. In other cases the ratio lies between zero and one. The quantity r is given by

$$r = \pm \sqrt{\frac{\text{explained variation}}{\text{total variation}}} = \pm \sqrt{\frac{\sum (Y_{\text{est}} - \bar{Y})^2}{\sum (Y - \bar{Y})^2}}$$

and varies between -1 and $+1$. The signs $\pm$ are used for positive linear correlation and negative linear correlation respectively. Note that r is a dimensionless quantity, i.e., it does not depend on the units employed.

The ideas which we have discussed with regard to linear curves fitting can be extended to non-linear curves.

CORRELATION
COEFFICIENT

NON-LINEAR



CHAPTER VIII
INPUT / OUTPUT

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Input at greater than 30 char/sec.	159
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The 9100A calculator can be interfaced with other equipment by using the connector on the rear. The calculator can both receive information and give information, and thereby control devices connected to it.

Any key on the keyboard can be activated from the rear connector to accomplish entry of data, execution of an operation or entry of a program step. Codes are entered digit serial-bit parallel. Operations and digits use a six bit code.

Information can be obtained from the display using signals available on the back connector of the 9100A. The information appears digit serial-bit parallel with the least significant digit of the displayed number first. The output section of this report explains the complete procedure.

Program steps stored in program memory and the operation codes for each keyboard execution are additional pieces of information available from the rear connector.

The FMT and PRINT keys are two keys on the calculator used for controlling external devices. The PRINT key causes a signal to appear on one line on the connector and is used to activate the 9120A printer. The FMT key similarly activates a line on the rear connector. The power of the FMT key is expanded by grounding another line on the rear connector. Each FMT operation must then be followed by another key to complete the cycle. For example, FMT-0 advances one line on the 9120A while FMT- $\uparrow$ and FMT- $\downarrow$ are used to move the pen on the 9125A X-Y plotter. FMT followed by any other key on the calculator is an instruction free to control some external device. After any FMT operation, the calculator waits for a signal from the external device before continuing.

The information rate is an important consideration when the calculator is part of a system. The various operations within the calculator are a time sequence of events which can be interrupted only at particular times. For this reason there is an upper limit to data entry determined by the internal string of operations required. For data entry there is a slow but simple technique and a faster but more involved method. The operations and their limitations are summarized below.

Type	Speed	Comment
Input data	Digit rate < 3 kc	No display above 30/sec
Output data	Digit rate < 2 kc	Least significant digit first
Input pgm. stp.	Entry rate < 2 kc	No display > 30/sec
Output pgm. stp.	Step rate < 2 kc	No display > 30/sec
Operation code during store pgm.	< 30/sec	Slow rate only
Operation code during execution	.03 ms<time<320 ms	Limited by execution time
FMT or PNT signals	Rate < 2 kc	From program
PNT operation	< 60/sec	X, Y, Z information

The length of cable over which the signals from calculator may be transmitted without buffering is limited to approximately 3 feet. Some of the 9100A signals are as short as 550 ns from an impedance of 2.2 k Ω , while others are D.C. As line length increases, the effects of common line capacitance between signal leads tends to obscure the desired signal. However, many feet of coaxial transmission line can be used when the 550 ns signals are buffered with emitter followers a foot or two from the calculator.

The inputs to the 9100A are subject to 500 ns sampling. The inductance of long wires may prevent maintaining adequate logic levels at the 9100A connector. Usually these problems can be solved by using a braided shield around the input wires as the ground reference.

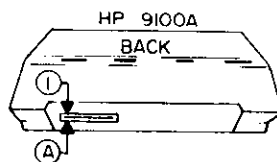
SIGNALS

The signals available on the 44 pin rear connector are listed below, each with a brief indication of its purpose and type - input or output. The running time of a program can be estimated by summing the execution time plus 0.2 ms for each step of the program.

	Signal	Brief Purpose	Type
Keyboard Oriented	NK20	Keyboard lines entry of codes	In
	NK21		In
	NK22		In
	NK23		In
	NK24		In
	NK25		In
	YKDN	Keyboard entry flag	In
	YRAD	Forces degrees	In
	YFLT	Forces fixed	In
	NFLT	Forces float	In
	YRUN	Forces program mode	In
	YINH	Disables keyboard	In
	ISTP	Indicates stop key pushed	Out
	NHLD	Holds at sync state	In
	NDGO	Passover sync state	In
Logic Oriented	YAFZ	Allow FMT types	In
	HFMT	Indicates FMT oper.	Out
	HOUT	Indicates PRINT oper.	Out
	F40B	Reg. of display & program with IOPF	Out
	F41B		Out
	F20B		Out
	F21B	Output of codes for numbers and operations	Out
	F22B		Out
	F23B		Out
	F24B		Out
	F25B		Out
	ITBB	Output code strobe	Out
	IADV	Timing strobe	Out
	ICHF	Indicates start of display	Out
	IOPF	Indicates new operation	Out
	YCOD	Used to look at character codes of the display	In
	YCHA	Used to look at decoded 7 segment codes of display	In
	YDSP	Forces exit from display	In
	YSKIC	Used to store program steps	In
	YGNPS	Used to look at stored program steps	In

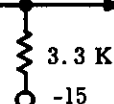
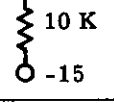
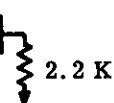
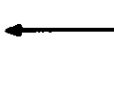
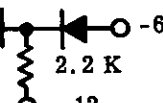
Plug Connections

NK25	A	1	NK24
NHLD	B	2	NK20
YRAD	C	3	NK21
YFLT	D	4	NK22
YRUN	E	5	NK23
HOUT	F	6	HFMT
F20B	H	7	F21B
F22B	J	8	F23B
F24B	K	9	F25B
NFLT	L	10	
ITBB	M	11	ISTP
IADV	N	12	ICHF
IOPF	P	13	F41B
YINH	R	14	F40B
YKDN	S	15	YCOD
YGNPS	T	16	YDSP
YSKIC	U	17	YCHA
YAFZ	V	18	NDGO
	W	19	SPARE 3
GND	X	20	END
SPARE 2	Y	21	SPARE 1
-15 V	Z	22	-15 V

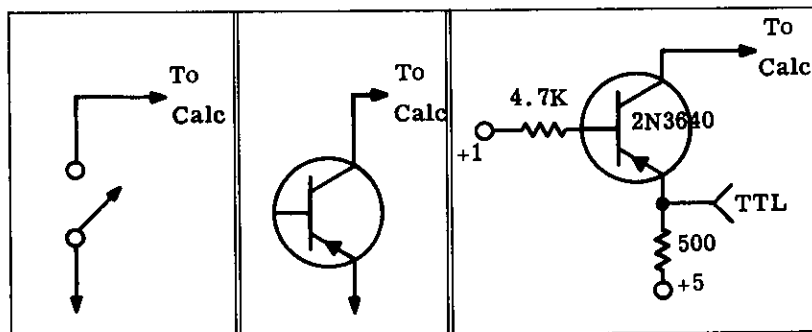


Plug Location

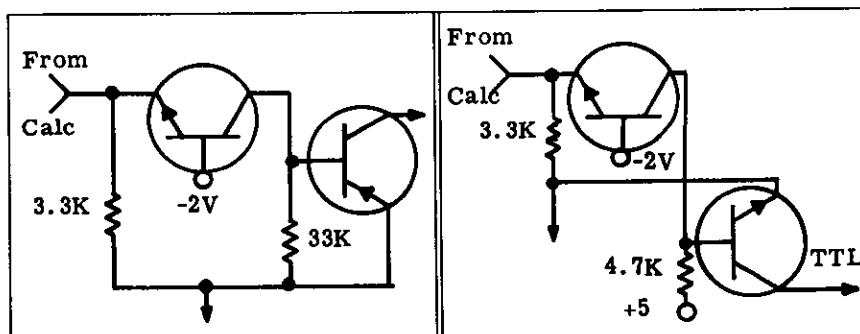
Logic Levels and
Equivalent Circuits
Seen Looking Into
the 9100A Calculator

Type	Boolean	Equivalent Circuit	
		Outside Calc.	Inside Calc.
Input Signal	0	$-1 < V < +10$ 	Signal 
	1	$-30 < V < -4$ 	Signal 
Output Signal	0		
	1		

EXAMPLE INPUT
DRIVING CIRCUITS



EXAMPLE OUTPUT
DETECTION CIRCUITS



POL	ABS	ARC
62	55	72
400	.03	.03
RCT	INT	HYP
66	64	67
400	.06	.03
RCL	EXP	SIN
61	74	70
.5	130	200
ACC-	LN	COS
63	65	73
5	70	200
ACC+	LOG	TAN
60	75	71
5	90	200

a	b	RDN
13	14	31
.25	.25	.45
c	d	RUP
16	17	22
.25	.25	.45
e	f	DN
12	15	25
.25	.25	.3
YTO	YE	UP
40	24	27
.03	.03	
XTO	XEY	
23	30	
.03	.25	.3

SQT	CHS	EEX	CLX
76	32	26	37
30	.03	.03	.12
÷	7	8	9
35	07	10	11
27	.03	.03	.03
x	4	5	6
36	04	05	06
22	.03	.03	.03
-	1	2	3
34	01	02	03
2	.03	.03	.03
+	0	.	PI
33	00	21	56
2	.03	.03	.1

CLR	IFL	SFL
20	43	54
1	3	.03
FMT	X<Y	PSE
42	52	57
.02	3	150
PNT	X=Y	STP
45	50	41
.03	3	.03
CNT	X>Y	END
47	53	46
	3	.2
GTO	STEP	
	44	51
.05	.03	.

Key
Code Octal
Max Time MS

Key Codes and Maximum Execution Times

Organized by Keyboard Position

Key Codes and
Maximum Execution
Times Organized
by Operation

Key	Octal Code	Max Time MS
.	21	.03
+	33	2
-	34	2
÷	35	27
x	36	22
0	00	.03
1	01	.03
2	02	.03
3	03	.03
4	04	.03
5	05	.03
6	06	.03
7	07	.03
8	10	.03
9	11	.03
a	13	.25
ABS	55	.03
ACC+	60	5
ACC-	63	5
ARC	72	.03
b	14	.25
c	16	.25
CHS	32	.03
CLR	20	1
CLX	37	.12
CNT	47	.05
COS	73	200
d	17	.25
DN	25	.3
e	12	.25
EEX	26	.03
END	46	.2

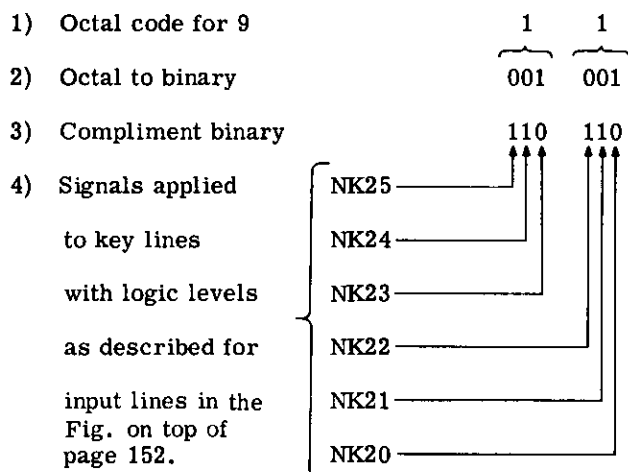
Key	Octal Code	Max Time MS
EXP	74	130
f	15	.25
FMT	42	.02
GTO	44	.03
HYP	67	.03
IFL	43	3
INT	64	.06
LN	65	70
LOG	75	90
PI	56	.1
PNT	45	.03
POL	62	400
PSE	57	150
RCL	61	.5
RCT	66	400
RDN	31	.45
RUP	22	.45
SFL	54	.03
SIN	70	200
SQT	76	30
STEP	51	-
STP	41	.03
TAN	71	200
UP	27	.3
X=Y	50	3
X<Y	52	3
X>Y	53	3
XKEY	30	.25
XTO	23	.03
YE	24	.03
YTO	40	.03
LN 10*	77	.1

Each operation requires 0.2 ms
more when executed in a program.

*Not on keyboard

The simplest way to enter information into the calculator is explained in the following sections and may be used providing the entry rate can be less than 30 ch/sec. This method is also used by the keyboard to enter keystrokes. Devices such as mechanical switches can be used for input because there is a built in delay which will allow 12 ms for contact bounce to end. Digits are entered in serial digit-parallel bit BCD with most significant digit first.

NK20-NK25 are six key lines on which binary key codes can be entered into the calculator externally. The desired key code is selected from the list of octal codes. These codes are the same as listed on the pull out card under the 9100A. To enter a particular code convert the octal code to binary and compliment it (change 1's to 0's and 0's to 1's). This code is applied to the input lines. NK25 is the most significant bit; NK20 is the least significant bit. Below is an example for entering the digit 9.



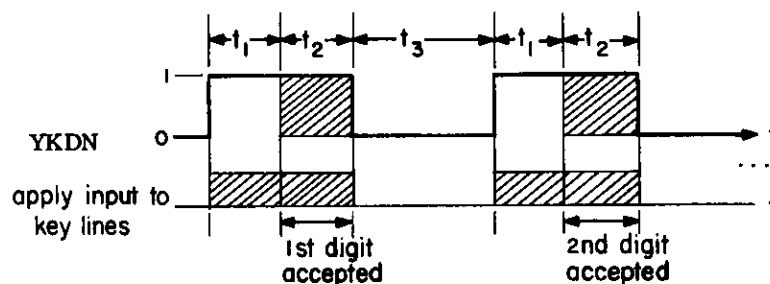
When the key code has been applied to NK25 - NK20, YKDN is used as a signal to enter the information. Since the keyboard also used NK 25 - NK20, these lines should see a high impedance when not being used externally.

YKDN is a signal used to tell the calculator that there is a code on the key lines. YKDN can be used only if the calculator is in display. Otherwise it does not cause an entry. When YKDN goes to a 1, there will be a 12 ms delay before the calculator can accept the code on the key lines. Some additional time is required to insure that the calculator has accepted the code. Additional time is required to reset the input for the next entry.

The simplest entry of digits is limited to 20 digits/sec in fixed point display and 30 digits/sec in floating point display and uses just YKDN and the key lines. All digit entries go to the X-register.

INPUT AT LESS THAN
30 CHAR/SEC

Digit Entry in Fixed Point Display



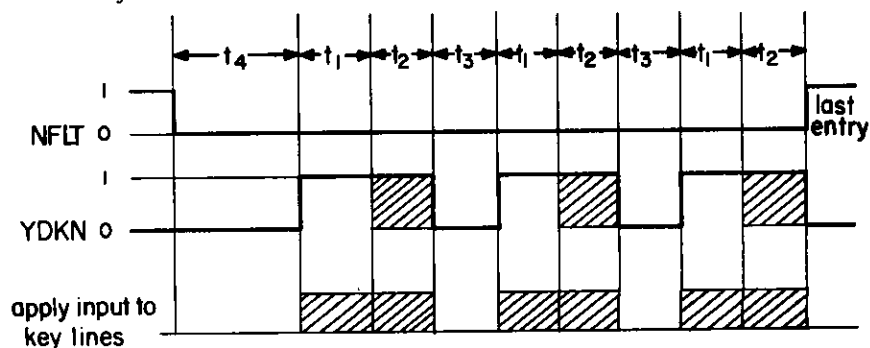
$t_1 = 12 \text{ ms}$ Contact bounce delay
 $t_2 \geq 20 \text{ ms}$ Data accepted
 $t_3 \geq 20 \text{ ms}$ Reset time

Two signals YFLT and NFLT allow external control of the FIXED-FLOAT switch.

FIXED-FLOAT set to	YFLT	NFLT	Display will be in
FIXED	1	1	FIXED
FLOAT	1	1	FLOAT
either	1 or 0	0	FLOAT
either	0	1	FIXED

Using NFLT to force the display to floating point, data can safely be entered up to 30 digits/sec. When the calculator is in floating point, it will check for data entry more often than in fixed point allowing faster data entry.

Digit Entry in Floating Point Display



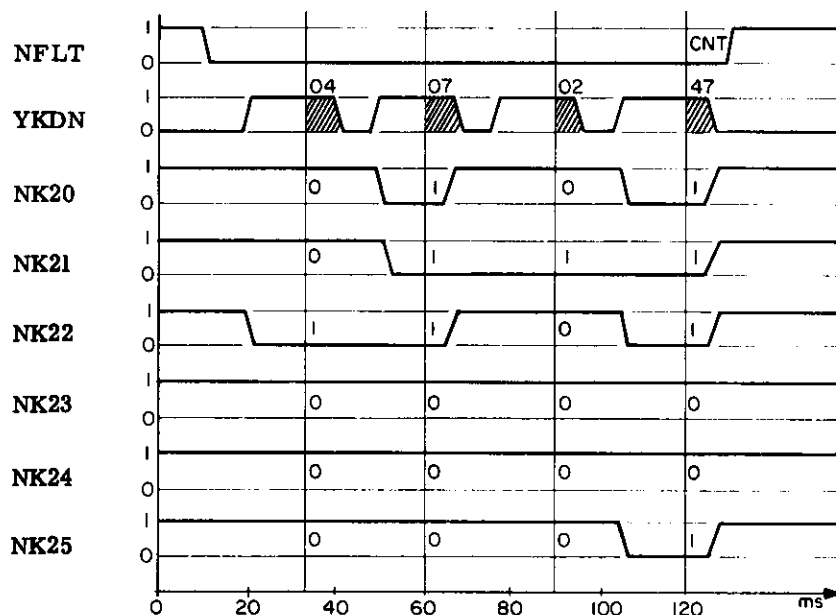
$t_4 \geq 20 \text{ ms}$ - initialization
 $t_2 \geq 10 \text{ ms}$ - accept data
 $t_1 = 12 \text{ ms}$ - delay
 $t_3 \geq 10 \text{ ms}$ - reset

As can be seen from the previous discussion, digit entry externally is directly analogous to digit entry from the keyboard. Information could be entered in floating point by entering in the correct sequence the codes for EEX and digit for the exponent or CHS for negative numbers.

Any key operation can be entered externally; however, before the next operation can be entered, the external device must wait for any additional execution time listed on page 154. This time is added to t_2 . For example, if the key entry were a COS function which requires 200 ms maximum time, t_3 would have to be at least 210 ms long in a floating point display.

ENTER 472 FROM DVM IN A PROGRAM

- 1) Assume that the calculator is in display either fixed or float.
- 2) Use a 4 bit scanner to select the most significant BCD digit (4) from the counter. Each digit is entered in order after being selected by the scanner. At the end of the entry the calculator can be started in a program by entering CONTINUE.



The calculator may be externally forced to the Program mode with YRUN.

YRUN	Calculator Enters
1	According to RUN-PRGM Switch
0	Program Steps

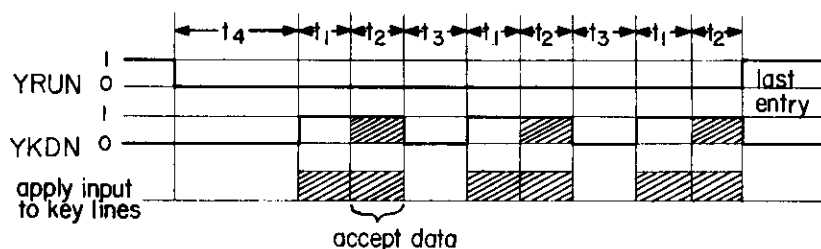
Program steps may be entered by following the timing below.

EXAMPLE

Example
Entering 472
CONTINUE

Use of YRUN

Entering Program Steps



$t_4 \geq 20$ ms - initialization

$t_1 = 12$ ms - delay

$t_2 \geq 10$ ms - accept data

$t_3 \geq 10$ ms - reset

The calculator may be forced to compute in degrees for circular functions by using YRAD.

Use of YRAD

YRAD	Circular Computations
1	According to DEG-RAD Switch
0	In Degrees

When a program calls for data from some external device, the calculator must signal this device to enter new data. There are some outputs called HOUT, ICHF, and HFMT which could be used for this purpose, and these signals are described in the output section of this report. For very simple systems it may be possible to wait long enough to know that the calculator has finished before entering the new number. In any case, the calculator must be displaying before any new digits can be entered.

Any time that NK20 - NK25 are being used externally, pushing a key down on keyboard may result in the wrong code being entered. A signal called YINH will prevent keyboard entry of all keys except stop and pause as shown below.

Use of YINH

YINH	Keyboard
1	All keys enabled
0	Only STOP and PAUSE enabled

An output called ISTP is tied to the stop key.

Use of ISTP

ISTP	Function
1	STOP key is down
0	STOP key not down

When other keys are disabled by an external device, ISTP provides an override. For example, the X-Y plotter uses YINH while moving. ISTP provides a means for the operator to lift the pen and return it to its zero position if an error causes the pen to move off scale.

The previous section describes a simple method of entering data, but the rate is slow, because time is required for keyboard delay and for display. This section describes how to avoid the display portion of the 9100A logic which allows up to 2000 entries/sec.

Two signals, called NHLD and NDGO, are used to allow external control of the synchronization of the display. Just prior to the formation of the least significant character of the X register, the calculator waits for a synchronization signal. Normally this signal is derived from the power line, but using NHLD and NDGO the time spent waiting may be externally controlled.

NHLD	NDGO	Display Will
1	1	Synchronize with power line
0	1	Stop at sync state in less than 20 ms
1 or 0	0	Not wait in sync state. If the display has been stopped when NDGO = 0, the logic will check for key entry before starting the display.

INPUT AT
GREATER
THAN
30/SEC

Use of NHLD
and NDGO

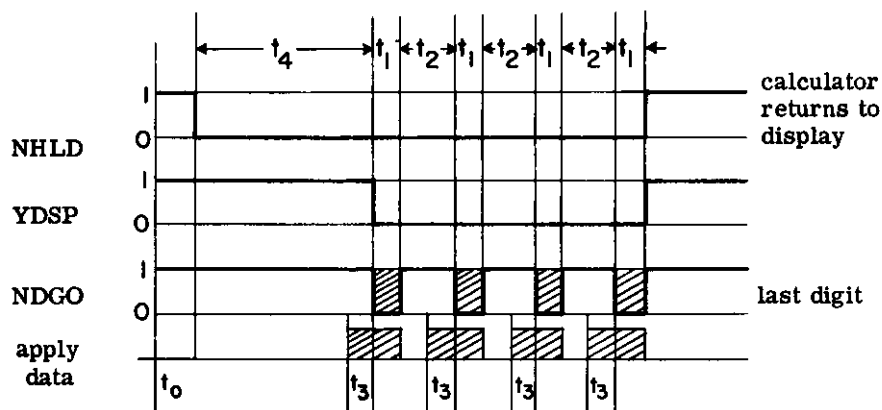
A signal called YDSP will simulate a key entry following the sync state. However, the 9100A will continue to repeat the operation coded on the NK20 - NK25 lines each time the calculator leaves the sync state and YDSP=0.

YDSP	9100A Operation
1	Normal display
0	a) Exit from display at sync state b) Execute code on NK20-NK25 according to RUN-PROG switch c) Repeat (b) each exit from sync state

Use of YDSP
Signal

Digit entry rate can be as high as 2000/sec by using NHLD, NDGO and YKDN. Before digits can be entered at this rate the calculator must be stopped in the sync state using NHLD. YDSP is then set to 0 making the 9100A ready for entries. Each time the NDGO is pulsed to 0, the key code on NK20-NK25 is entered. RUN-PROGRAM switch must be set to run. The timing involved in digit entry at this rate is described below.

Data Entry at 2000 Digits/Sec Rate



- $t_1 > 10 \mu s$ and $< 30 \mu s$ - digit entry
- $t_2 \geq 500 \mu s$ - wait
- $t_4 \geq 10 ms$ if calculator is in display at t_0 or time until 550 ns pulse on ICHF if calculator is not in display at t_0
- $t_3 \geq 0$ - data lead time

The calculator may be started in a program after any entry by entering the code 47 for CONTINUE as the last code.

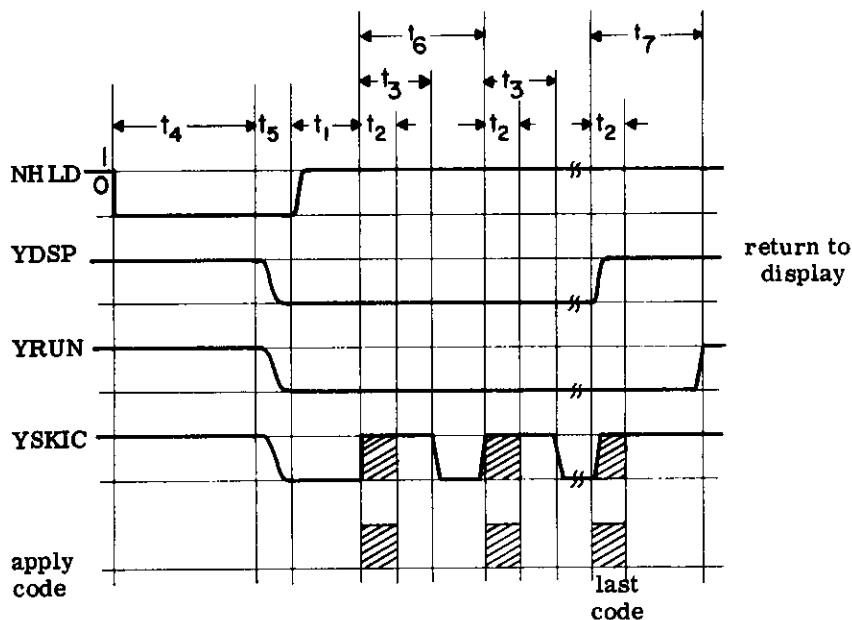
Program steps may be entered so that the X register displays the steps or so that there is no change in the X register. Program steps displayed in X register may be entered using timing described above with the addition that YRUN = 0 during the entry period or the RUN-PROGRAM switch can be set to PROGRAM. Entering program steps without changing the X register requires the use of the YSKIC signal.

GRAM

Use of YSKIC

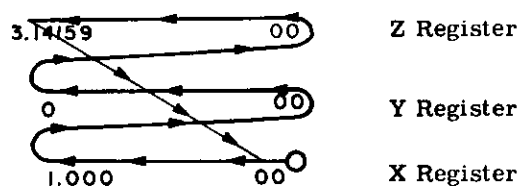
YSKIC	When Entering Programs, 9100A Will
1	Operate as normal
0	Stop just before storing the next program step. When YSKIC goes to 1, NK20-NK25 code will be stored as a prog. step

Storing program steps without changing the X register follows the timing below. However, the signal strobe for a store operation is now YSKIC rather than NDGO.



- | | |
|------------------------------------|-----------------------|
| $t_1 = \geq 250 \mu s$ | - initialize |
| $t_2 = 10 \mu s$ | - code accepted |
| $t_3 > 10 \mu s$ and $< 150 \mu s$ | - data strobe time |
| $t_4 \geq 20 ms$ | - initialize |
| $t_5 > 0$ | - race avoidance time |
| $t_6 \geq 500 \mu s$ | - repetition time |
| $t_7 \geq 50 \mu s$ | - race avoidance time |

The display monitor is a cathode ray tube which displays the characters with an electrostatically deflected electron beam. There are three registers displayed from bottom to top called the X, Y and Z registers. The registers are traced one digit at a time from right to left starting with X then Y then Z.



This order of display was chosen to allow round up of the last place of a register in fixed point.

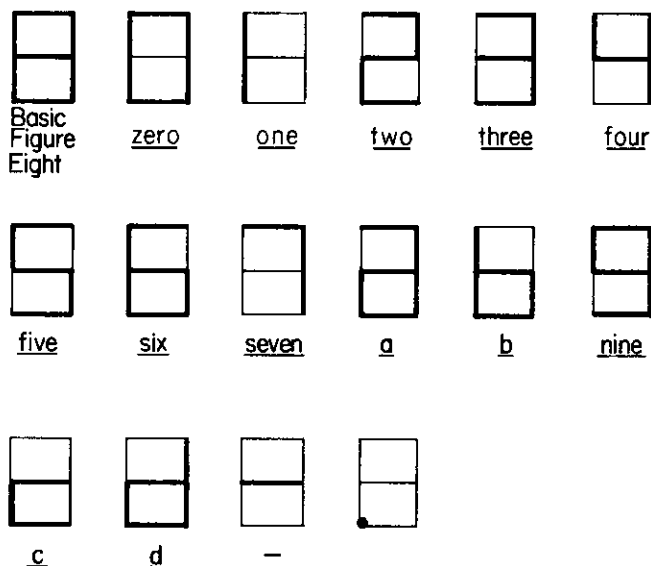
Entering Program
Steps up to 2000/
sec Without Changing
the X Register

OUTPUT OF DISPLAYED INFORMATION

Formation of
Display

7-Segment Character Formation

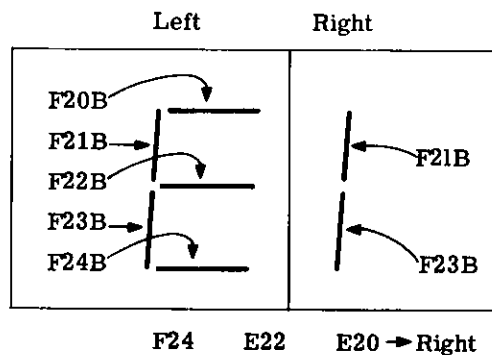
The normal display is a field of figure eight patterns. The characters are formed by blanking (turning off electron beam) during various segments of this figure eight pattern.



Each figure eight pattern is formed from a left and a right hand part. The left hand part is a figure E pattern with 5 segments and the right hand part is a line with 2 segments. The codes for the characters are BCD and are available from the interface connector. The left and right hand character codes are also available and are used to generate characters for the 9120A printer.

The codes from the display appear on outputs F20B-F25B of the interface plug.

Character Codes in Display



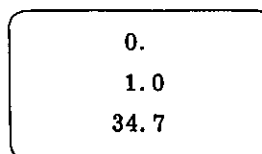
If signal = 1 then that segment is blanked for that character

Use of YCOD and YCHA

YCOD and YCHA are used to read codes from the display by causing the calculator to repeat an output every 1.6 μ s.

YCOD	YCHA	ITBB and Output Behavior
1	1	One ITBB and character code for each half of the characters
1	0	Display will stop. ITBB gives repeated left or right segment code
0	1	Display will stop. ITBB gives repeated character codes for each half

For example, if the display reads



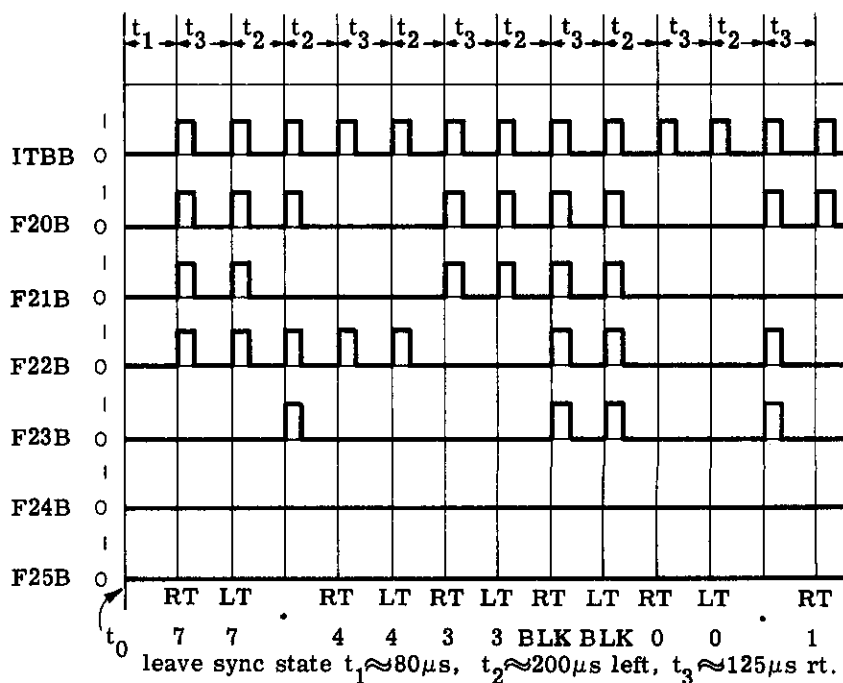
FLOAT ☐ FIXED

Decimal wheel set to 1.

after the sync state the first digits will appear as indicated below if YCOD and YCHA are equal to 1.

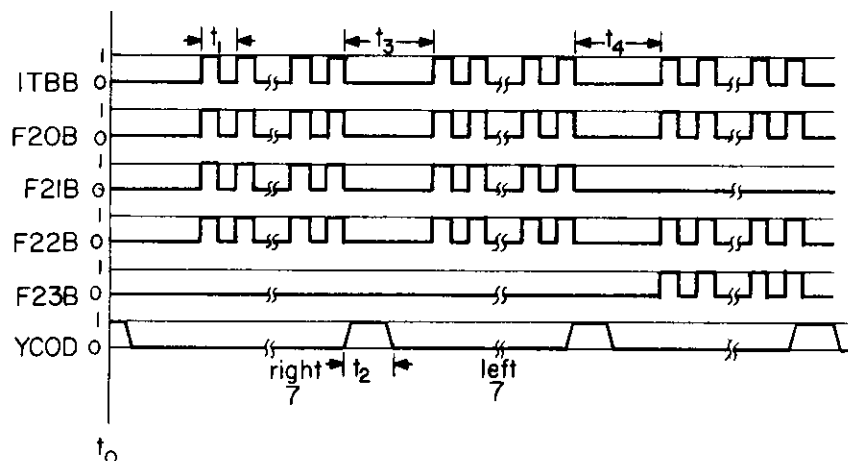
There is some delay between ITBB outputs which is the time required to draw character segments on the CRT.

Display Output When YCHA= YCOD = 1



Note that the decimal point has only one half. To read one code for every character just take every other code starting with the first.

Use of YCOD to Report Character Code



$t_1 = 1.6 \mu s$ - repetition
 $t_2 \geq 10 \mu s$ & $< t_3$ - strobe
 $t_4 \approx 200 \mu s$ left
 $t_3 \approx 125 \mu s$ right

} character formation

YCHA causes the left and right half codes to appear rather than the character codes. The timing is the same with YCOD replaced by YCHA. The first ITBB of each string of repeated outputs strobes the character code. The remaining ITBB signals strobe left or right half codes.

When a digit output occurs, F40B and F41B tells the register from which it came by the code:

Register Coding with F40B and F41B

F40B	F41B	When in Display Calculator State is
1	1	Waiting in sync state
0	0	X register
1	0	Y register
0	1	Z register

Note that F40B and F41B provide a means of determining when the calculator has reached the sync state but this indication is only valid when the calculator is in display, i. e., not performing operations.

Use of ICHF

It is possible to determine when the calculator returns to the display from any operation or program by monitoring the signal ICHF.

ICHF	Meaning
0	Normal value
1	550 ns pulse indicating return to sync state

Meaning of HOUT

HOUT is a signal which indicates a PRINT signal either from keyboard or program. HOUT normally is used to trigger the 9120A Printer but it could be used to signal some other device if desired, if the printer is not attached.

HOUT	Meaning
0	Normal value
1	550 ns pulse, PRINT operation

HOUT will appear about 20 μ s before ICHF.

Meaning of HFMT

HFMT can be used to signal external devices if HOUT is already being used. HFMT indicates a FMT signal either from keyboard or program.

HFMT	Meaning
0	Normal value
1	550 ns pulse, FMT operation

Effects of YAFZ

YAFZ is a signal which allows external modification of the FMT instruction such that each FMT operation must be followed by a second key step. Any key may follow FMT so that 63 different commands are possible. In order to differentiate between these FMT operations externally, an ITBB will follow HFMT to strobe the 6 bit code for the second key step of the FMT operation on line F20B-F25B.

YAFZ	Type of FMT
1	Single step, always FMT - 0
0	Two step, FMT - Designation

LADV is a 550 ns pulse in display which may be used to ready an external device for a new code strobed by ITBB. IADV occurs at least once about $8\mu s$ before each ITBB in display and once following the end of the Z register.

It is desirable to know when a function is performed from a program, because the external device must enter a CONTINUE code if the program is to continue. IOPF is a 550 ns signal which indicates the start of any new operation whether from program or from keyboard. When combined with F41B, IOPF gives the desired information.

Use of IOPF to
Detect Program Operations

IOPF	F41B	Meaning
0	0	Normal value
0	1	No meaning
1	0	550 ns pulse Start of keyboard operation
1	1	550 ns pulse Start of program operation

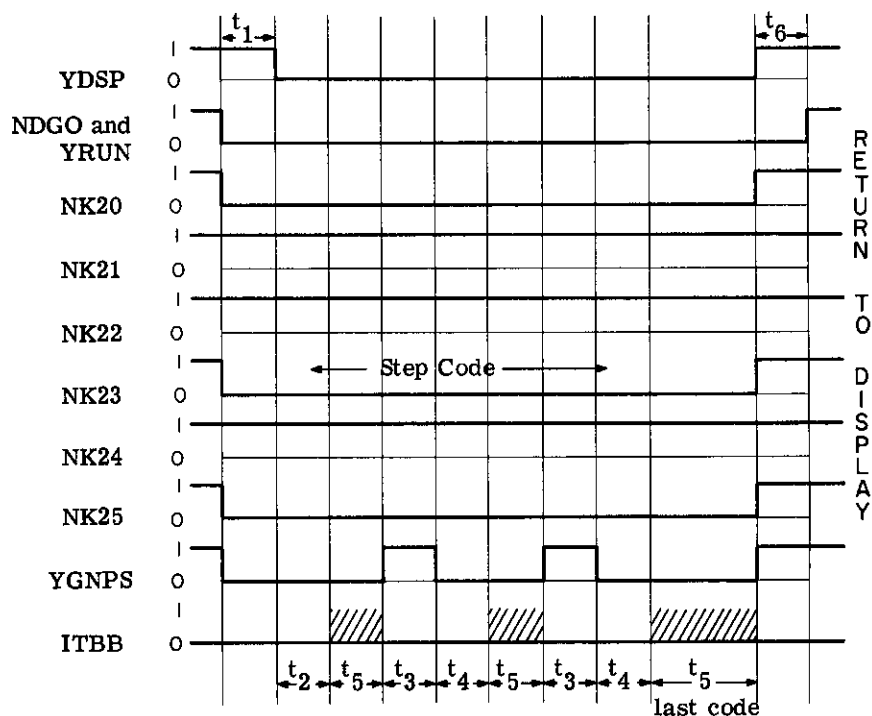
Stored program steps may be externally read from the calculator using a signal called YGNPS in conjunction with some other signals. The basic idea is to enter the single step operation with the calculator in program while bypassing the display and use the function of single step which retrieves program steps from memory. YGNPS causes the program outputs to repeat.

Function of
YGNPS

YGNPS	When Entering Program Steps, 9100A Will Output
0	One ITBB and code for each step entered or one ITBB and stored code if (step) is entered.
1	Repeated ITBB and stored code if (step) is entered.

YGNPS is used according to timing on the next page. Updating the program counter is automatic after each strobe period t_3 .

Reading Program steps Using YGNPS

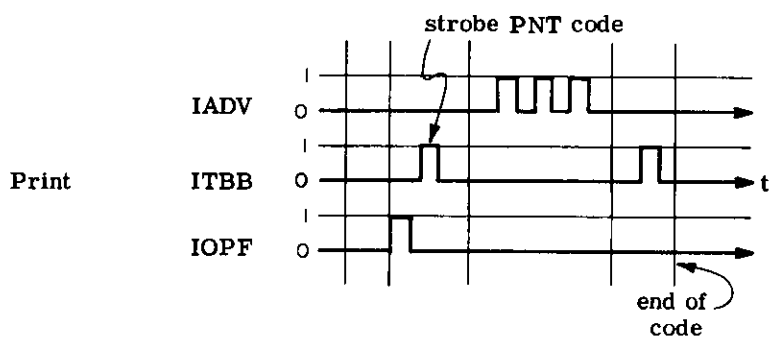
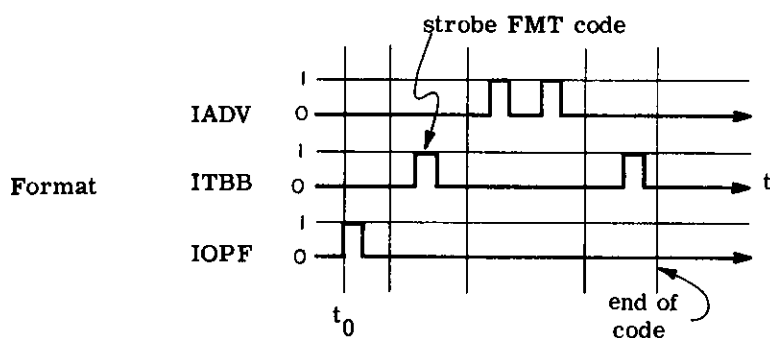
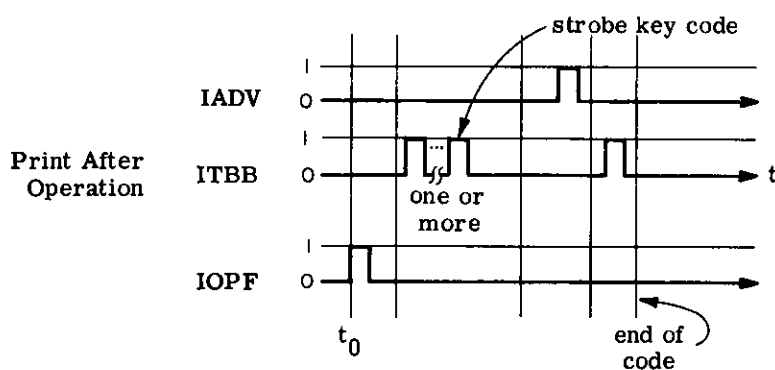


- $t_1 \geq 2 \mu s$ - race avoidance
- $t_2 < 6 ms$ - initialize and fetch
- $t_3 \geq 10 \mu s \text{ \& } < 150 \mu s$ - strobe
- $t_4 < 250 \mu s$ - fetch
- $t_5 = \text{as long as desired}$ - read program step
- $t_6 > 300 \mu s$ - race avoidance

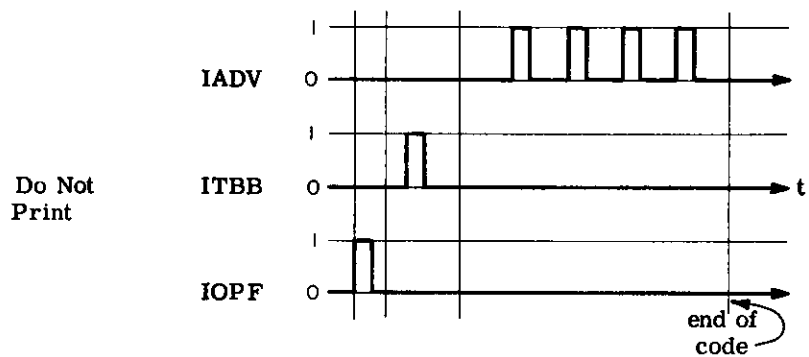
Specification of Print After Operation

The key codes are listed on page 153 for all the operations in the 9100A. IOPF indicates the beginning of each new operation. A string of ITBB and IADV pulses follow IOPF which may be interpreted to tell when the key code is on the F20B-F25B outputs, when a PNT operation was chosen or when the operation is classed as a print after operation. Print after operation includes a group of functions which would cause a print of the display if the calculator were to behave as a printing calculator. On the next page are groups of functions which should and should not cause a print depending on the RUN-PROGRAM switch. Using these signals, a printer can be constructed to convert the calculator to a printing calculator with labels for all operations.

	9100A in RUN	9100A in PROGRAM
Print After Operation	Digits following YTO XTO, YE and all else except below	All
No Print	CLX CLD 0 9 EEY CHS FMT GTO	None



PULSE COUNTING CODES



Any output occurring after end of code but before a pulse on ICHF is to be ignored.

The key code for any key included in print after operation group will be on F20B-F25B simultaneously with the last ITBB pulse before the IADV pulses.



